Intermediary Leverage Cycles and Financial Stability

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Abstract

We present a theory of financial intermediary leverage cycles within a dynamic model of the macroeconomy. Intermediaries face risk based funding constraints that give rise to procyclical leverage and a procyclical share of intermediated credit. The pricing of risk varies as a function of intermediary leverage, and asset return exposures to intermediary leverage shocks earn a positive risk premium. Relative to an economy with constant leverage, financial intermediaries generate higher consumption growth and lower consumption volatility in normal times, at the cost of endogenous systemic financial risk. The severity of systemic crisis depends on two state variables: intermediaries' leverage and net worth. Regulations that tighten funding constraints affect the systemic risk-return trade-off by lowering the likelihood of systemic crises at the cost of higher pricing of risk.

JEL classification: E02, E32, G00, G28

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1 Introduction

The financial crisis of 2007-09 highlighted the central role that financial intermediaries play in the propagation of fundamental shocks. In this paper, we develop a general equilibrium model in which the endogenous leverage cycle of financial intermediaries creates propagation and amplification of fundamental shocks. The model features endogenous solvency risk of the financial sector, allowing us to study the impact of prudential policies on the trade-off between systemwide distress and the pricing of risk during normal times.

We build on the emerging literature[1] on dynamic macroeconomic models with financial intermediaries by assuming capital regulation is risk based, implying that institutions have to hold equity in proportion to the riskiness of their total assets. Our model gives rise to the procyclical leverage behavior documented by Adrian and Shin (2010, 2014), and the procyclicality of intermediated credit documented by Adrian, Colla, and Shin (2012). Furthermore, prices of risk fluctuate as a function of intermediary leverage, and the price of risk of leverage is positive, both features that have been as shown in asset pricing tests by Adrian, Moench, and Shin (2010, 2014) and Adrian, Etula, and Muir (2014).

In our theory, financial intermediaries have two roles. While both households and intermediaries can own existing firms' capital, intermediaries have access to a better capital creation technology, capturing financial institutions' ability to allocate capital more efficiently and monitor borrowers. The second role of intermediaries is to provide risk bearing capacity by accumulating inside equity. Intermediaries' ability to bear risk fluctuates over time due to the risk sensitive nature of their funding constraint.

The combination of costly adjustments to the real capital stock and the risk based leverage constraint lead to the intermediary leverage cycle, which translates into an endogenous amplification of shocks.[2] When adverse shocks to intermediary balance sheets are sufficiently large, intermediaries experience systemic solvency risk and need to restructure. We assume that such systemic distress occurs when intermediaries' net worth falls below a threshold. Intermediaries deleverage by writing down debt, imposing losses on households. Whether systemic financial crises are be-

[1] Brunnermeier and Sannikov (2011, 2014), Gertler and Kiyotaki (2012), Gertler, Kiyotaki, and Queralto (2012), He and Krishnamurthy (2012, 2013) all have recently proposed equilibrium theories with a financial sector.

[2] While fundamental shocks are assumed to be homoskedastic, equilibrium asset prices and equilibrium consumption growth exhibit stochastic volatility.

nign or generate large consumption losses depends on the severity of the shocks, the leverage of intermediaries, and their relative net worth.

Our model gives rise to the "volatility paradox" of Brunnermeier and Sannikov (2014): Times of low volatility tend to be associated with a buildup of leverage, which increases forward-looking systemic risk. We also study the systemic risk-return trade-off: Low prices of risk today tend to be associated with larger forward-looking systemic risk measures, suggesting that measures of asset price valuations are useful indicators for systemic risk assessments. The pricing of risk, in turn, is tightly linked to the Lagrange multiplier on intermediaries' risk based leverage constraint, which determines their effective risk aversion.

Our theory provides a conceptual framework for financial stability policies. In this paper, we focus on capital regulation.[3] We show that households' welfare dependence on the capital constraint is inversely U-shaped: very loose constraints generate excessive risk taking of intermediaries relative to household preferences, while very tight funding constraints inhibit intermediaries' risk taking and investment. This trade-off maps closely into the debate on optimal regulation.[4]

In the benchmark model with a constant intermediary leverage constraint, the equilibrium growth of investment, price of capital, and the risk-free rate are constant. Fluctuations in output of the benchmark economy are entirely due to productivity shocks, as output is fully insulated from liquidity shocks. In contrast, in the model with a risk based funding constraint, liquidity shocks spill over to real activity, and productivity shocks are amplified.

This paper is related to several strands of the literature. Geanakoplos (2003) and Fostel and Geanakoplos (2008) show that leverage cycles can cause contagion and issuance rationing in a general equilibrium model with heterogeneous agents, incomplete markets, and endogenous collateral. Brunnermeier and Pedersen (2009) further show that market liquidity and traders' access to funding are co-dependent, leading to liquidity spirals. Our model differs from that of Fostel and Geanakoplos (2008) as our asset markets are dynamically complete and debt contracts are not collateralized. The leverage cycle in our model comes from the risk-based leverage constraint of the financial intermediaries and is intimately related to the funding liquidity of Brunnermeier and

[3]The literature considering (macro)prudential policies in dynamic equilibrium models is small but growing rapidly, see Goodhart, Kashyap, Tsomocos, and Vardoulakis (2012), Angelini, Neri, and Panetta (2011), Angeloni and Faia (2013), Korinek (2011), Bianchi and Mendoza (2011), and Nuño and Thomas (2012) for complementary work.

[4]It should be noted that these results rely on our assumption that intermediaries finance themselves only in the public debt market, thus violating the necessary assumptions for the Modigliani and Miller (1958) capital structure irrelevance result. While the impact of prudential regulation would be less pronounced if intermediaries were able to issue equity, any positive cost of equity issuance would preserve the systemic risk-return trade-off.

Pedersen (2009). Unlike their model, however, the funding liquidity that matters in our setup is that of the financial intermediaries, not of speculative traders.

This paper is also related to studies of amplification in models of the macroeconomy. The seminal paper in this literature is Bernanke and Gertler (1989), which shows that the condition of borrowers' balance sheets is a source of output dynamics. Net worth increases during economic upturns, increasing investment and amplifying the upturn, while the opposite dynamics hold in a downturn. Kiyotaki and Moore (1997) show that small shocks can be amplified by credit restrictions, giving rise to large output fluctuations. Instead of focusing on financial frictions in the demand for credit as Bernanke and Gertler and Kiyotaki and Moore do, our theory focuses on frictions in the supply of credit. Another important distinction is that the intermediaries in our economy face leverage constraints that depend on current volatility, which give rise to procyclical leverage. In contrast, the leverage constraints of Kiyotaki and Moore are state independent and lead to countercyclical leverage.

Gertler, Kiyotaki, and Queralto (2012) and Gertler and Kiyotaki (2012) extend the accelerator mechanism of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) to financial intermediaries. Gertler, Kiyotaki, and Queralto (2012) consider a model in which financial intermediaries can issue outside equity and short-term debt, making intermediary risk exposure an endogenous choice. Gertler and Kiyotaki (2012) further extend the model to allow for household liquidity shocks as in Diamond and Dybvig (1983). While these models are similar in spirit to our work, our model is more parsimonious in nature and allows for endogenous defaultable debt. We can thus investigate the creation of systemic default and the effectiveness of macroprudential policy in mitigating these risks. Furthermore, our model generates procyclical leverage and a procyclical share of intermediated credit.

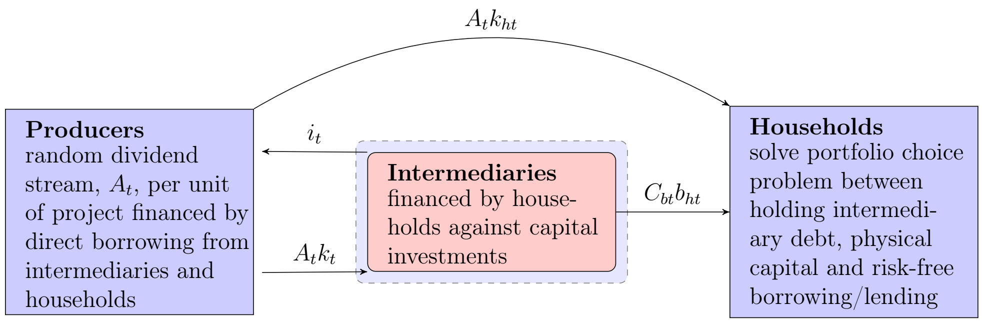
Our theory is closely related to the work of He and Krishnamurthy (2012, 2013) and Brunnermeier and Sannikov (2011, 2014), who explicitly introduce a financial sector into dynamic models of the macroeconomy. While our setup shares many conceptual and technical features of this work, our points of departure are empirically motivated. We allow households to invest via financial intermediaries as well as directly in the capital stock, a feature strongly supported by the data, which gives rise to important substitution effects between directly granted and intermediated credit. In the setup of He and Krishnamurthy, investment is always intermediated. Furthermore, our model features procyclical intermediary leverage, while theirs is countercyclical. Finally, systemic risk of the intermediary sector is at the heart of our analysis, while He and Krishnamurthy and Brunnermeier and Sannikov focus primarily on the amplification of shocks. In fact, in the set-up of He and Krishnamurthy, the financial sector is only constrained in times of crises. Thus, the consumption-CAPM holds during normal times, and intermediary wealth enters the pricing kernel in times of crises only. In contrast, in our approach, intermediary state variables (wealth and leverage) always enter into the pricing kernel, with the price of risk of leverage positive in all states of the economy, while the price of risk of output fluctuates generically between positive and negative values.

Our theory qualitatively matches stylized facts about the intermediary leverage cycle. These stylized facts rely on the cyclical behavior of mark-to-market leverage, and mark-to-market equity, following Adrian and Shin (2010, 2014), Adrian, Colla, and Shin (2012), and Adrian, Moench, and Shin (2010, 2014). In our model, as well as the models of He and Krishnamurthy and Brunnermeier and Sannikov, intermediary equity is non-traded. Instead, it is determined as the difference between the market value of assets of the intermediary and the market value of intermediary debt, making mark-to-market equity the appropriate empirical counterpart. Furthermore, in practice, market market value of equity captures the value of intangible assets that are not carried on the balance sheet of financial institutions. Adrian, Moench, and Shin (2014) conduct asset pricing tests using mark-to-market leverage and market leverage, and mark-to-market equity and market equity, and find that the mark-to-market measures fare better empirically.

The interactions between the households, the financial intermediaries, and the productive sector lead to a highly nonlinear system. We consider the nonlinearity a desirable feature, as the model is able to capture strong amplification effects. Our theory features both endogenous risk amplification (where fundamental volatility is amplified as in Danielsson, Shin, and Zigrand (2011)), as well as the creation of endogenous systemic risk.

In our theory, equilibrium dynamics are functions of two intermediary state variables: their leverage and their wealth. In contrast, in other equilibrium models with heterogenous agents, the relevant state variables are typically only wealth shares, not leverage. For example, in Rampini and Viswanathan (2012), the second variable is household wealth. In Dumas (1989) and Wang (1996), the state variables are aggregate output and the ratio of the marginal utilities of the two types of agents.

Figure 1. Economy Structure



2 A Model

We consider a single consumption good economy, with the unique consumption good at time $t > 0$ used as the numeraire. There are three types of agents in the economy: producers, financial intermediaries and households. We abstract from modeling the decisions of the producers and focus instead on the interaction between the intermediary sector and the households. The basic structure of the economy is represented in Figure 1.

2.1 Production

There is an "AK" production technology that produces $Y_t = A_t K_t$ units of output at each time t. The stochastic productivity of capital $\{A_t = e^{a_t}\}_{t \geq 0}$ follows a geometric diffusion process of the form

$$da_t = \bar{a}dt + \sigma_a dZ_{at},$$

with $\{Z_{at}\}_{0 \leq t < +\infty}$ a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each unit of capital in the economy depreciates at a rate λ_k, so that the capital stock in the economy evolves as

$$dK_t = (I_t - \lambda_k) K_t dt,$$

where I_t is the reinvestment rate per unit of capital in place. Thus, output in the economy evolves according to

$$dY_t = \left(I_t - \lambda_k + \bar{a} + \frac{\sigma_a^2}{2}\right) Y_t dt + \sigma_a Y_t dZ_{at}.$$

Notice that the quantity $A_t K_t$ corresponds to the "efficiency" capital of Brunnermeier and Sannikov (2014), with a constant productivity rate of 1. There is a fully liquid market for physical capital, in which both the financial intermediaries and the households are allowed to participate. To keep the economy scale-invariant, we denote by $p_{kt} A_t$ the price of one unit of capital at time t in terms of the consumption good.

2.2 Households

There is a unit mass of risk-averse, infinitely lived households in the economy. We assume that the households are identical, so that the equilibrium outcomes are determined by the decisions of the representative household. Households, however, are exposed to a preference shock, modeled as a change-of-measure variable in the household's utility function. This reduced-form approach allows us to remain agnostic as to the exact source of this second shock: With this specification, it can arise either from time-variation in the households' risk aversion, time preference, or beliefs. In particular, we assume that there is a household which evaluates different consumption paths $\{c_t\}_{t\geq 0}$ according to

$$\mathbb{E}\left[\int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t dt\right],$$

where ρ_h is the subjective time discount of the representative household, and c_t is the consumption at time t. Here, $\exp(-\xi_t)$ is the Radon-Nikodym derivative of the measure induced by households' time-varying preferences or beliefs with respect to the physical measure. For simplicity, we assume that $\{\xi_t\}_{t\geq 0}$ evolves as a Brownian motion, correlated with the productivity shock, Z_{at}:

$$d\xi_t = \sigma_\xi \rho_{\xi,a} dZ_{at} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} dZ_{\xi t},$$

where $\{Z_{\xi t}\}$ is a standard Brownian motion of $(\Omega, \mathcal{F}_t, \mathbb{P})$, independent of Z_{at}. In the current setting, with households constrained in their portfolio allocation, $\exp(-\xi_t)$ can be interpreted as a time-varying liquidity preference shock, as in Diamond and Dybvig (1983), Allen and Gale

(1994), and Holmström and Tirole (1998) or as a time-varying shock to the preference for early resolution of uncertainty, as in Bhamra, Kuehn, and Strebulaev (2010b,a). In particular, when the households receive a positive $d\xi_t$ shock, their effective discount rate increases, leading to a higher demand for liquidity. Including non-zero correlation in the model provides more flexibility in the correlation structure of equilibrium asset returns and thus provides an additional channel for amplification. In our simulations, we set this correlation $\rho_{\xi,a}$ to zero to focus on the intermediaries' role in amplifying shocks.

The households finance their consumption through holdings of physical capital, holdings of risky intermediary debt, and short-term risk-free borrowing and lending. The households are less productive users of capital than intermediaries; in particular, the households do not have access to the investment technology. Thus, the physical capital k_{ht} held by households evolves according to

$$dk_{ht} = -\lambda_k k_{ht} dt.$$

Each unit of capital owned by the household produces A_t units of output, so the total return to one unit of household wealth invested in capital is

$$dR_{kt} = \underbrace{\frac{A_t k_{ht}}{k_{ht} p_{kt} A_t} dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(k_{ht} p_{kt} A_t\right)}{k_{ht} p_{kt} A_t}}_{\text{capital gains}} \equiv \mu_{Rk,t} dt + \sigma_{ka,t} dZ_{at} + \sigma_{k\xi,t} dZ_{\xi t}.$$

In addition to direct capital investment, the households can invest in risky intermediary debt. To keep the balance sheet structure of the financial institutions time-invariant, we assume that the bonds mature at a constant rate λ_b, so that the time t probability of a bond maturing before time $t + dt$ is $\lambda_b dt$.[5] Thus, the risky debt holdings b_{ht} of households follow

$$db_{ht} = \left(\beta_t - \lambda_b\right) b_{ht} dt,$$

where β_t is the issuance rate of new debt. The bonds pay a floating coupon $C_{bt} A_t$ until maturity, with the coupon payment determined in equilibrium to clear the risky bond market. Similarly to capital, risky bonds are liquidly traded, with the price of a unit of intermediary debt at time t in terms of the consumption good given by $p_{bt} A_t$. Hence, the total return from one unit of household

[5] This corresponds to an infinite-horizon version of the "stationary balance sheet" assumption of Leland and Toft (1996). Allowing for bonds with a finite maturity gives rise to the possibility of default by financial intermediaries.

wealth invested in risky debt is

$$dR_{bt} = \underbrace{\frac{\left(C_{bt} + \lambda_b - \beta_t p_{bt}\right) A_t b_{ht}}{b_{ht} p_{bt} A_t}}_{\text{dividend-price ratio}} dt + \underbrace{\frac{d\left(b_{ht} p_{bt} A_t\right)}{b_{ht} p_{bt} A_t}}_{\text{capital gains}} \equiv \mu_{Rb,t} dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi t}.$$

Finally, we assume that the households face no-shorting constraints, so that $k_{ht} \geq 0$ and $b_{ht} \geq 0$. Thus, the households solve

$$\max_{\{c_t, \pi_{kt}, \pi_{bt}\}} \mathbb{E}\left[\int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t dt\right], \tag{2.1}$$

subject to the no-shorting constraints and household wealth evolution

$$dw_{ht} = r_{ft} w_{ht} + \pi_{kt} w_{ht} \left(dR_{kt} - r_{ft} dt\right) + \pi_{bt} w_{ht} \left(dR_{bt} - r_{ft} dt\right) - c_t dt, \tag{2.2}$$

where π_{kt} and π_{bt} are the fractions of household wealth invested in the risky capital and risky intermediary debt, respectively. We have the following result.

Lemma 2.1. *The household's optimal consumption choice satisfies*

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

In the unconstrained region, the household's optimal portfolio choice is given by

$$\begin{bmatrix} \pi_{kt} \\ \pi_{bt} \end{bmatrix} = \left(\begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix} \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mu_{Rk,t} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{bmatrix}$$
$$- \sigma_\xi \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}^{-1} \begin{bmatrix} \rho_{\xi,a} \\ \sqrt{1 - \rho_{\xi,a}^2} \end{bmatrix}.$$

Proof. See Appendix A.1. □

The household with the liquidity preference shocks chooses consumption as a log-utility investor but with a lower rate of discount. The optimal portfolio choice of the household, on the other hand, also includes an intratemporal hedging component for variations in the liquidity shock, $\exp(-\xi_t)$. Since intermediary debt is locally risk-less, however, households do not self-insure against intermediary default. Appendix A.1 also provides the optimal portfolio choice in the case when the household is constrained. In our simulations, the household never becomes constrained as the intermediary wealth never reaches zero. The presence of the liquidity shock induces households to hold both types of financial claims which is in contrast to the solution with only productivity shocks when households either invest in intermediary liabilities or in the capital stock.

2.3 Financial Intermediaries

There is a unit mass of infinitely lived financial intermediaries in the economy. As with the households, we assume that all financial intermediaries are identical and therefore equilibrium outcomes are determined by the behavior of the representative intermediary. We abstract from modeling the dividend payment decision ("consumption") of the intermediary sector and consider the intermediary sector to be a technology. The profits of the intermediaries are instead split between retained earnings and coupon payments to bondholders.

Financial intermediaries create new capital through capital investment. Denote by k_t the physical capital held by the representative intermediary at time t and by $i_t A_t$ the investment rate per unit of capital. Then the stock of capital held by the representative intermediary evolves according to

$$dk_t = (\Phi(i_t) - \lambda_k)\, k_t dt.$$

Here, $\Phi(\cdot)$ reflects the costs of (dis)investment. We assume that $\Phi(0) = 0$, so in the absence of new investment, capital depreciates at the economy-wide rate λ_k. Notice that the above formulation implies that costs of adjusting capital are higher in economies with a higher level of capital productivity, corresponding to the intuition that more developed economies are more specialized. We follow Brunnermeier and Sannikov (2014) in assuming that investment carries quadratic adjustment costs, so that Φ has the form

$$\Phi(i_t) = \phi_0 \left(\sqrt{1 + \phi_1 i_t} - 1 \right),$$

for positive constants ϕ_0 and ϕ_1. Quadratic adjustment costs capture the empirical regularity that new investments in physical capital are incrementally more expensive for larger investments (see e.g. Hayashi, 1982).

Each unit of capital owned by the intermediary produces $A_t(1 - i_t)$ units of output net of investment. As a result, the total return from one unit of intermediary capital invested in physical capital is given by

$$dr_{kt} = dR_{kt} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}} \right) dt,$$

so that, compared to the households, the financial intermediaries earn an extra return to holding firm capital to compensate them for the cost of investment. This extra return is partially passed on to the households as coupon payments on the intermediaries' debt.

Financial intermediaries serve two functions in our economy. First, they are more efficient users of productive capital and generate new investment. Second, they own equity that provides risk-bearing capacity, potentially absorbing aggregate risk, which benefits households. In particular, without intermediaries' debt issuance in the model, households would be unable to hedge their exposure to liquidity shocks. Compare this with the notion of intermediation of He and Krishnamurthy (2012, 2013). In their model, intermediaries provide households with access the risky investment technology: Without the intermediary sector, the households can only invest at the risk-free rate. Instead, in their setup, the households enter into a profit-sharing agreement with the intermediary, with the profits distributed according to the initial wealth contributions. This precludes intermediary default and prevents household preference shocks from being transmitted to the real economy. While in both our model and the model of He and Krishnamurthy the presence of intermediaries improves the risk-sharing ability of households, the nature of risk-sharing in our model is different as intermediaries provide insurance against both productivity and preference shocks. Furthermore, our model also generates a procyclical share of intermediated credit to the productive sector.

The intermediaries finance their investment in new capital projects by issuing risky floating coupon bonds to the households and through retained earnings. We assume that intermediary borrowing is restricted by a risk-based capital constraint, similar to the value at risk (*VaR*) constraint of Danielsson, Shin, and Zigrand (2011). In particular, we assume that

$$\alpha \sqrt{\frac{1}{dt} \left\langle k_t d\left(p_{kt} A_t\right)\right\rangle^2} \leq w_t, \tag{2.3}$$

where $\langle \cdot \rangle^2$ is the quadratic variation operator. That is, the intermediaries are restricted to retain enough equity to cover a certain fraction of losses on their assets. Unlike a traditional *VaR* constraint, this does not keep the volatility of intermediary equity constant, leaving the intermediary sector exposed to solvency risk. The risk-based capital constraint implies a time-varying leverage constraint θ_t, defined by

$$\theta_t = \frac{p_{kt} A_t k_t}{w_t} \leq \frac{1}{\alpha \sqrt{\frac{1}{dt} \left\langle \frac{d(p_{kt} A_t)}{p_{kt} A_t} \right\rangle^2}}.$$

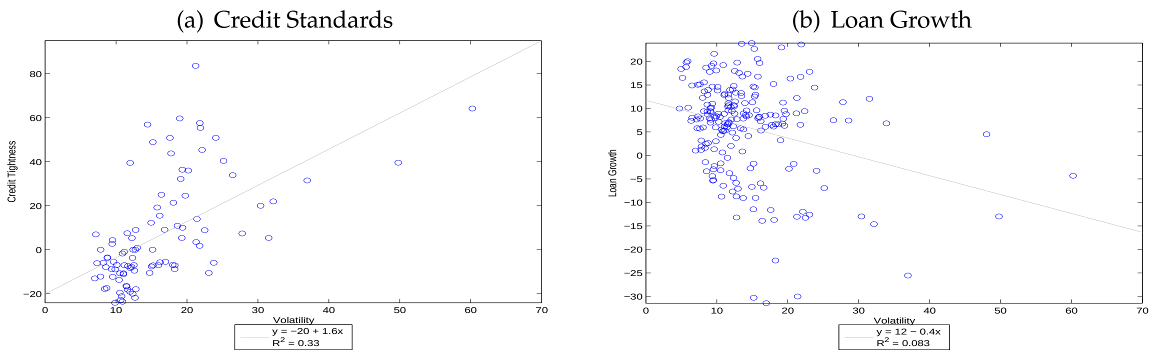
The per-dollar total *VaR* of assets is thus negatively related to intermediary leverage, as documented in Adrian and Shin (2014). The parameter α determines how much equity the intermediary has to hold for each dollar of asset volatility. We interpret this parameter α as a policy parameter that is pinned down by regulation. α determines the tightness of risk based capital requirements, similar to the capital requirements coordinated by the Basel Committee on Banking Supervision.

Assumption (2.3) is key to generating the procyclical behavior of leverage and countercyclical behavior of intermediary mark-to-market equity that we observe empirically. While risk-based capital constraints can microfounded be as optimal contracts in the presence of moral hazard concerns (see Adrian and Shin (2014) for a static setting and Nuño and Thomas (2012) for a dynamic setting), we consider the constraints faced by our intermediaries as imposed by regulation.[6]

The risk based capital constraint of intermediaries is directly related to the way in which financial intermediaries manage market risk. Trading operations of major banks – most of which are undertaken in the security broker-dealer subsidiaries – are managed by allocating equity in relation to the *VaR* of trading assets. Constraint (2.3) directly captures such behavior. Banking books, on the other hand, are managed either according to credit risk models, or using historical cost accounting rules with loss provisioning. Although the risk-based capital constraint does not directly capture these features of commercial banks' risk management, empirical evidence suggests that the risk based funding constraint is a good behavioral assumption for bank lending. Panel (a) of Figure 2 shows that the tightness of credit supply conditions reported by the Senior Loan Officer Survey of the Federal Reserve increases following increases in realized market volatility, and Panel (b) shows that loan growth at commercial banks decreases following increases in realized market volatility. A higher level of asset volatility is thus associated with tighter lending conditions of commercial banks.

[6]The risk based capital constraint is closely related to a Value at Risk constraint. Value at Risk constraints originated from risk management practices of investment banks in the 1980s, and were subsequently adopted in the Basel II capital framework, which was adopted by investment banks in the U.S. in 2004.

Figure 2. Market Volatility and Credit Supply. Scatter plots and best linear fit between the credit tightening indicator from the Board of Governors of the Federal Reserve System Senior Loan Officer Opinion Survey and the realized S&P 500 volatility over the previous quarter (Panel a) and between the annualized growth rate of commercial bank loans to nonbank corporate business and lagged realized volatility (Panel b). Source: Haver DLX.



We assume that the financial intermediaries are myopic and maximize an instantaneous mean-variance objective of wealth w_t, as in He and Krishnamurthy (2012),

$$\max_{\theta_t, i_t} \quad \mathbb{E}_t\left[\frac{dw_t}{w_t}\right] - \frac{\gamma}{2}\mathbb{V}_t\left[\frac{dw_t}{w_t}\right], \tag{2.4}$$

subject to the dynamic intermediary budget constraint

$$dw_t = \theta_t w_t dr_{kt} - (1-\theta_t)\, w_t dR_{bt}. \tag{2.5}$$

and the risk-based capital constraint constraint (2.3). Here, γ measures the degree of risk-aversion of the representative intermediary; when γ is close to zero, the intermediary is almost risk-neutral and chooses its portfolio each period to maximize the expected instantaneous growth rate. We have the following result.

Lemma 2.2. *The representative financial intermediary optimally invests in new projects at rate*

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

For nearly risk-neutral intermediaries (γ close to 0), the risk-based capital constraint binds, and the shadow cost of increased leverage is

$$\zeta_t = \left(\mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right) - r_{ft}\right) - \left(\mu_{Rb,t} - r_{ft}\right) + \gamma\left[\sigma_{ba,t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right) + \sigma_{b\xi,t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\right]$$

$$-\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}}\left[\left(\sigma_{ka,t}-\sigma_{ba,t}\right)^2+\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)^2\right].$$

When intermediaries are not capital constrained, the optimal leverage choice of the intermediary is given by

$$\theta_t=\frac{\left(\mu_{Rk,t}+\Phi\left(i_t\right)-\frac{i_t}{p_{kt}}-r_{ft}\right)-\left(\mu_{Rb,t}-r_{ft}\right)}{\gamma\left[\left(\sigma_{ka,t}-\sigma_{ba,t}\right)^2+\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)^2\right]}+\frac{\left[\sigma_{ba,t}\left(\sigma_{ka,t}-\sigma_{ba,t}\right)+\sigma_{b\xi,t}\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)\right]}{\left[\left(\sigma_{ka,t}-\sigma_{ba,t}\right)^2+\left(\sigma_{k\xi,t}-\sigma_{b\xi,t}\right)^2\right]}.$$

Proof. See Appendix A.2. □

In this paper, we assume that the intermediaries' risk aversion is sufficiently low to make the risk-based capital constraint always bind.[7] This simplifying assumption captures the empirically-documented short-termism of financial intermediaries. In contrast, the intermediaries of Brunnermeier and Sannikov (2011, 2014) manage their leverage so as to make sure that they have a big enough buffer to make their debt instantaneously risk free. The intertemporal risk management of the intermediary is then driving their effective risk aversion, pinning down their leverage and balance sheet growth. In our approach, when γ is close to 0, intermediaries leverage to the maximum, with their effective risk aversion determined by the Lagrange multiplier ζ_t on their capital constraint.

The risk-based capital constraint (2.3) does not prevent intermediary wealth from becoming negative as the instantaneous volatility of intermediary equity is not constant. To prevent this counterfactual outcome, we follow Black and Cox (1976) and assume that the intermediary is restructured when its equity falls below an exogenously specified threshold, $\bar{\omega}p_{kt}A_tK_t$. We allow the distress boundary $\bar{\omega}p_{kt}A_tK_t$ to grow with the scale of the economy, so that the intermediary can never outgrow the possibility of distress. When the intermediary is restructured, the management of the intermediary changes. The new management defaults of the debt of the previous intermediary, reducing leverage to $\underline{\theta}$, but maintains the same level of capital as before. The inside equity of the new intermediary is thus

$$w_{\tau_D^+}=\bar{\omega}\frac{\theta_{\tau_D}}{\underline{\theta}}p_{k\tau_D}A_{\tau_D}K_{\tau_D},$$

[7]In a companion paper, we show that our results hold qualitatively in a setting where the leverage constraints binds only sometimes, see Adrian and Boyarchenko (2013).

where τ_D is the first hitting time of the restructuring region

$$\tau_D = \inf_{t \geq 0} \{ w_t \leq \bar{\omega} p_{kt} A_t K_t \} .$$

We define the term structure of distress risk to be

$$\delta_t (T) = \mathbb{P} (\tau_D \leq T | (w_t, \theta_t)) .$$

Here, $\delta_t (T)$ is the time t probability of default occurring before time T. Since the fundamental shocks in the economy are Brownian, and all the agents in the economy have perfect information, the local distress risk is zero. Intermediary restructuring is a systemic risk as it affects the representative intermediary in the economy. In our simulations, we use parameter values for $\bar{\omega}$ that are positive (not zero), thus viewing intermediaries distress as a restructuring event.

2.4 Equilibrium

Definition 2.1. *An equilibrium in this economy is a set of price processes* $\{p_{kt}, p_{bt}, C_{bt}\}_{t \geq 0}$*, a set of household decisions* $\{\pi_{kt}, \pi_{bt}, c_t\}_{t \geq 0}$*, and a set of intermediary decisions* $\{\beta_t, i_t, \theta_t\}_{t \geq 0}$ *such that:*

1. *Taking the price processes and the intermediary decisions as given, the household's choices solve the household's optimization problem (2.1), subject to the household budget constraint (2.2).*

2. *Taking the price processes and the household decisions as given, the intermediary's choices solve the intermediary optimization problem (2.4), subject to the intermediary wealth evolution (2.5) and the risk-based capital constraint (2.3).*

3. *The capital market clears:*

$$K_t = k_t + k_{ht}.$$

4. *The risky bond market clears:*

$$b_t = b_{ht}.$$

5. *The risk-free debt market clears:*

$$w_{ht} = p_{kt}A_t k_{ht} + p_{bt}A_t b_{ht}.$$

6. *The goods market clears:*

$$c_t = A_t \left(K_t - i_t k_t\right).$$

Notice that the bond markets' clearing conditions imply

$$p_{kt}A_t K_t = w_{ht} + w_t.$$

Notice also that the aggregate capital in the economy evolves as

$$dK_t = -\lambda_k K_t dt + \Phi\left(i_t\right) k_t dt = \left(\Phi\left(i_t\right)\frac{k_t}{K_t} - \lambda_k\right) K_t dt.$$

We solve for the equilibrium in terms of two state variables: the leverage of the financial intermediaries, θ_t, and the fraction of wealth in the economy owned by the intermediaries

$$\omega_t = \frac{w_t}{w_t + w_{ht}} = \frac{w_t}{p_{kt}A_t K_t}.$$

By construction, the household belief shocks are expectation-neutral, and thus their level is not a state variable in the economy. Similarly, we have defined prices in the economy to scale with the level of productivity, A_t, so productivity itself is not a state variable in the scaled version of the economy. We represent the evolution of the state variables as

$$\begin{aligned}\frac{d\omega_t}{\omega_t} &= \mu_{\omega t}dt + \sigma_{\omega a,t}dZ_{at} + \sigma_{\omega\xi,t}dZ_{\xi t}\\ \frac{d\theta_t}{\theta_t} &= \mu_{\theta t}dt + \sigma_{\theta a,t}dZ_{at} + \sigma_{\theta\xi,t}dZ_{\xi t}.\end{aligned}$$

We can then express all the other equilibrium quantities, including the drift and volatility of these state variables, in terms of the state variables, and the sensitivities $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$ of the return to holding capital to output and liquidity shocks. The following Lemma summarizes the properties of the solution.

Lemma 2.3. *In equilibrium, the expected excess return on capital and risky intermediary debt, as well as the expected return on intermediary equity, the risk-free rate, and the volatility of intermediary equity and intermediary debt, can be expressed as linear combinations of the exposure of capital returns to productivity shocks, $\sigma_{ka,t}$, and liquidity shocks, $\sigma_{k\xi,t}$, with the coefficients non-linear functions of the state (θ_t, ω_t). The exposure of capital returns to productivity shocks, $\sigma_{ka,t}$, and liquidity shocks, $\sigma_{k\xi,t}$, are given, respectively, by*

$$\sigma_{k\xi,t} = -\sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \sigma_{ka,t}^2}$$

$$\sigma_{ka,t} = \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \left(1 + \frac{1-\omega_t}{\omega_t \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right)}\right).$$

Proof. See Appendix B. □

Notice that the negative root determines the exposure of capital to the household liquidity shock, $\sigma_{k\xi,t}$. Intuitively, when the household experiences a liquidity shock, such that $dZ_{\xi t} > 0$, the household's discount rate increases, causing a reallocation to capital and away from intermediary debt, decreasing the return to holding capital. The details of the solution are relegated to Appendix B.

3 Model Simulation

We illustrate the equilibrium outcomes in our economy by focusing on the empirical facts from previous literature that can be replicated by our model. We show that the model generates amplification and propagation of shocks as the slow evolution of endogenous volatility generates a leverage cycle that impacts the pricing of risk and credit extension. We simulate 10000 paths of the economy using parameters in Table 1. Each path is simulated at a monthly frequency, with the economy running for 80 years to match the time since the Great Depression. In Tables 2-4, we report the mean and the median regression coefficients, together with the fifth and ninety-fifth percentile outcomes from the simulations. We plot the median path in the corresponding Figures 3 and 5-6.

We plot the impulse responses of variables of interest to a one standard deviation shock to productivity, $\sigma_a dZ_{at}$, and a one standard deviation shock to the households' preference for liquidity, $\sigma_\xi dZ_{\xi t}$ in Figure 4. We compute the percent deviation of the path following a shock from the path that the economy would have taken if no shock occurred. In particular, we first compute

Table 1: Parameters used in simulations

Parameter	Notation	Value
Expected growth rate of productivity	$\bar{a}$	0.0651
Volatility of growth rate of productivity	σ_a	0.388
Volatility of liquidity shocks	σ_ξ	0.0388
Discount rate of intermediaries	ρ	0.06
Effective discount rate of households	$\rho_h - \sigma_\xi^2/2$	0.05
Fixed cost of capital adjustment	ϕ_0	0.1
	ϕ_1	20
Depreciation rate of capital	λ_k	0.03

the benchmark path of variables of interest without any shocks, but still subject to the endogenous drift of the state variables in the economy. That is, the benchmark path is calculated setting $dZ_{a,t+s} = dZ_{\xi,t+s} = 0$ for $s \geq 0$. Next, we compute the corresponding "shocked" path given the initial shock $dZ_{at} = -1$ (for the impulse response of a shock to productivity) or $dZ_{\xi t} = 1$ (for the impulse response of a shock to liquidity) but setting all future realizations of shocks to zero ($dZ_{a,t+s} = dZ_{\xi,t+s} = 0$ for $s > 0$). We calculate the impulse response function as the percentage difference between the shocked and the benchmark paths. This computation is meant to mimic a deviation from steady state computation that is typically plotted in impulse response functions in linear non-stochastic models.

The intuition for the two shocks is as follows. A negative productivity shock impairs the asset side of intermediary balance sheets, inducing them to sell capital to the household sector. Because households value the capital relatively less than intermediaries do (as households cannot generate new capital), the price of capital declines, increasing intermediary leverage and reducing their relative net worth. The higher leverage can only be supported by lower equilibrium return volatility. In contrast, a liquidity shock leads households to sell intermediary debt and capital, leading to a relative wealth gain for the intermediaries, and a decline in their leverage. Equilibrium return volatility increases, as a smaller fraction of total wealth of the economy is allocated to risky assets.

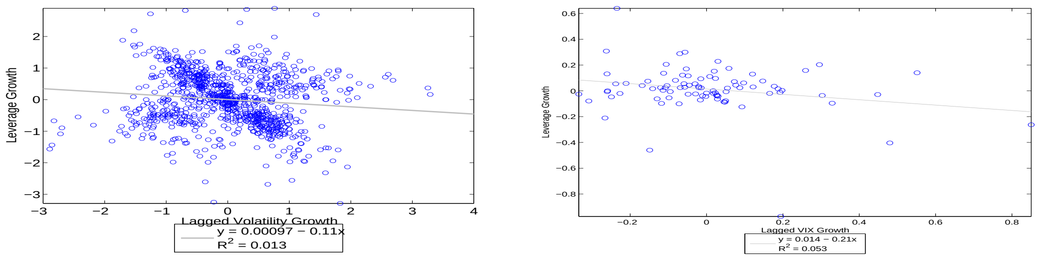
3.1 Balance sheet evolution

We begin by studying the equilibrium evolution of the intermediary balance sheet and credit creation by intermediaries. From the *VaR* constraint, we have

$$\alpha^{-2}\theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2.$$

Figure 3. Intermediary Leverage and Lagged Volatility Growth

(a) Simulated (b) Data



NOTES: The relationship between the growth rate of leverage of financial institutions and the lagged growth rate of implied volatility. Right panel: quarterly growth of broker-dealer leverage (y-axis) versus lagged quarterly growth of the Chicago Board Options Exchange (CBOE) market volatility index (VIX) (x-axis); left panel: quarterly growth of intermediary leverage, θ_t, (y-axis) versus lagged quarterly growth of capital return volatility, $\sqrt{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}}$, ($x$-axis) for a representative path. Data on broker-dealer leverage are from Flow of Funds Table L.129. Data from the model is simulated using parameters in Table 1 at a monthly frequency for 80 years.

Thus, the riskiness of the return to holding capital increases as intermediary leverage decreases. We plot the theoretical and the empirical trade-off between leverage growth and volatility in Figure 3. Higher levels of the VIX tend to precede declines in broker-dealer leverage (right panel). In the model, this translates into a negative relationship between the lagged growth rate of asset return volatility and intermediary leverage growth (left panel). The negative relationship between broker-dealer leverage and the VIX is further investigated in Adrian and Shin (2010, 2014).[8] While the evidence from Figure 3 is from broker-dealers, it also has an empirical counterpart for the banking book. As discussed earlier, the lending standards of banks vary tightly with the VIX, indicating that new lending of commercial banks is highly correlated with measures of market volatility.

Table 2 reports the coefficients and the R^2 of the regression of broker-dealer leverage growth on lagged growth in implied volatility in the data (first column) and in the model. The model generates a consistently negative relationship between leverage growth and lagged return volatility, even for the paths with the largest (least negative) linear coefficients (last column).

[8]While Adrian and Shin (2010) show that fluctuations in primary dealer repo—which they show to be a proxy for fluctuations in broker-dealer leverage—tend to forecast movements in the VIX, Figure 3 shows that higher levels of the VIX precede declines in broker-dealer leverage. We use the lagged VIX as the VIX is implied volatility and hence a forward-looking measure (though the negative relationship also holds for contemporaneous VIX). Adrian and Shin (2014) use the *VaR* data of major securities broker-dealers to show a negative association between broker-dealer leverage growth and the *VaRs* of the broker-dealers.

	Data	Mean	5%	Median	95%
β_0	0.014	0.000	-0.003	0.000	0.003
β_1	-0.208	-0.105	-0.187	-0.104	-0.025
R^2	0.053	0.013	0.001	0.011	0.035

Table 2: Intermediary Leverage and Lagged Volatility Growth

NOTES: The relationship between the growth rate of leverage of financial institutions and the lagged growth rate of implied volatility. The "Data" column reports the coefficients estimated using broker-dealer leverage growth as the dependent variable and the growth rate of the Chicago Board Options Exchange (CBOE) market volatility index (VIX) as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized growth rate of leverage, θ_t, of the intermediaries as the dependent variable, and growth rate of total volatility of the return on capital, $\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on broker-dealer leverage are from Flow of Funds Table L.129.

To understand the economic mechanism that generates this, consider the response in intermediary leverage, return to capital volatility and its individual components—$\sigma_{ka,t}$ and $\sigma_{k\xi,t}$—and expected returns in response to a productivity shock (i.e. a temporary one standard deviation decline in productivity). Panels (a) and (b) of Figure 4 show that a one standard deviation decline in the productivity of capital decreases both $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$ in the short-run. This leads to a decrease in total volatility (Panel d), relaxing the *VaR* constraint, thus allowing intermediary leverage to increase (Panel c). In the long run, while the loading of the return to capital on the productivity shock $\sigma_{ka,t}$ overshoots that for the benchmark path, the loading on the liquidity shock $\sigma_{k\xi,t}$ remains depressed relative to the benchmark path. Total volatility of the return to holding capital thus remains depressed even in the long run (Panel d), and the intermediaries are able to take on more leverage (Panel c).

The impulse response functions follow a similar logic in the case of a one standard deviation shock to the households' liquidity preference, but in opposite direction. A liquidity shock increases volatility in the short run, decreasing intermediaries' risk bearing capacity (Panels c and d). This effect persists even in the long run as $\sigma_{k\xi,t}$ remains elevated.

These impulse response functions illustrate the amplification and propagation embedded in the model due to the interplay of endogenous leverage and intermediaries' ability to take risk. Even one-off shocks have persistent effects on the behavior of intermediary balance sheets, as the shock propagates through the persistent impact of equilibrium volatility on intermediaries' ability to take risk. In addition, Figure 4 illustrates the amplification of underlying shocks through the leverage cycle. Shocks to liquidity and productivity are followed by a persistent deviation of

	Data	Mean	5%	Median	95%
β_0	-0.071	-0.112	-0.203	-0.108	-0.040
β_1	0.756	0.434	0.190	0.433	0.680
R^2	0.460	0.048	0.009	0.045	0.101

Table 3: Procyclicality of Intermediated Credit

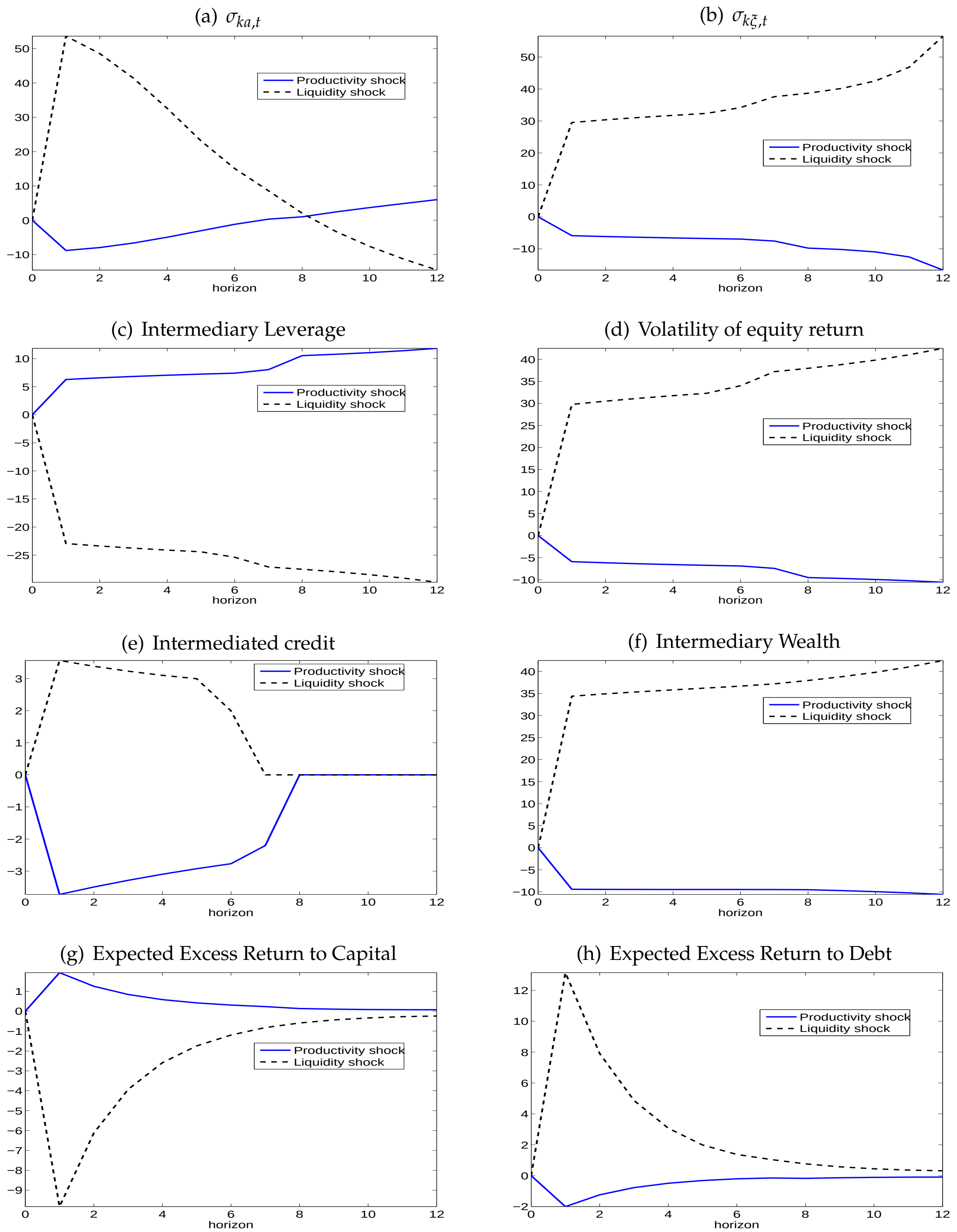
NOTES: The relationship between total credit in the economy and the amount of credit extended through the financial intermediary sector. The "Data" column reports the coefficients estimated using the growth rate of credit extended by financial intermediaries to the non-financial corporate sector as the dependent variable, and the growth rate of total credit to the non-financial corporate sector as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized growth rate of capital held by intermediaries, k_t, as the dependent variable, and the growth rate of total capital in the economy, K_t, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on total credit to the nonfinancial corporate sector and the share of intermediated finance are from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets are from Flow of Funds Table L.129.

leverage, volatility, and wealth from the benchmark path. The persistent and amplified impacts on volatility and leverage also result in persistent impacts on risk premia, wealth accumulation and credit supply, which we discuss next.

Turning to the provision of intermediated credit and risk premia in the economy, Panel (e) of Figure 4 plots the response of the fraction of intermediated credit to a one standard deviation shock to productivity and to households' preference for liquidity. A negative productivity shock increases the expected excess return to holding capital (Panel g of Figure 4). This increases households' willingness to hold capital directly, reducing the fraction of credit intermediated through the financial system. In the long run, higher intermediary leverage raises the expected excess return to holding intermediary debt, and the households re-optimize their portfolio holdings to hold more intermediary debt and less productive capital. This leads the intermediated credit in the model to be procyclical: the fraction of credit through the financial intermediaries has a strong positive relationship with total credit extended to the productive sector. The coefficients of the corresponding regression for both the model and the data (column 1) are reported in Table 3, with the linear coefficient remaining positive even for extreme paths, with the linear coefficient remaining positive even for extreme paths.

Intermediated credit has the opposite response to a shock to households' preference to liquidity: a positive shock to ξ increases households' rate of time discount, increasing the expected return to holding intermediary debt (Panel h of Figure 4) and lowering the expected return to holding capital. Thus, households reallocate their portfolios toward holding more debt, and relaxing constraints on credit intermediation to the productive sector (Panel e). In the long run, as inter-

Figure 4. Impulse response functions



NOTES: Effect of a $-\sigma_a$ shock to productivity (solid line) and a σ_ξ shock to household discount rate (dashed line) on return volatility, intermediary balance sheets and expected excess returns to debt and capital.

mediaries reduce leverage and build more equity (Panels c and f of Figure 4), capital once again becomes an attractive investment for households, and the fraction of intermediated credit returns to the benchmark path. Expected returns to capital and debt also revert (Panels g and h).

Figure 5 plots the growth of the share of intermediated credit as a function of total credit growth, showing the strong positive relationship in the model and the data. This positive relationship has been previously documented in Adrian, Colla, and Shin (2012) and shows the procyclical nature of intermediated finance. The middle panel of Figure 5 shows the procyclical nature of the leverage of financial intermediaries. Leverage tends to expand when balance sheets grow, a fact that has been documented by Adrian and Shin (2010) for the broker-dealer sector, by Adrian, Colla, and Shin (2012) for the commercial banking sector, and Adrian and Shin (2014) for the largest bank holding companies. The lower panel shows that the procyclical leverage translates into countercyclical equity growth, both in the data and in the model. We should note that the procyclical leverage of financial intermediaries is closely tied to the risk-based capital constraint. In contrast, previous literature has found it challenging to generate this feature and in fact exhibits countercyclical leverage (see e. g. Brunnermeier and Sannikov, 2011, 2014; He and Krishnamurthy, 2012, 2013; Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997; Gertler and Kiyotaki, 2012; Gertler, Kiyotaki, and Queralto, 2012).

3.2 Equilibrium pricing kernel

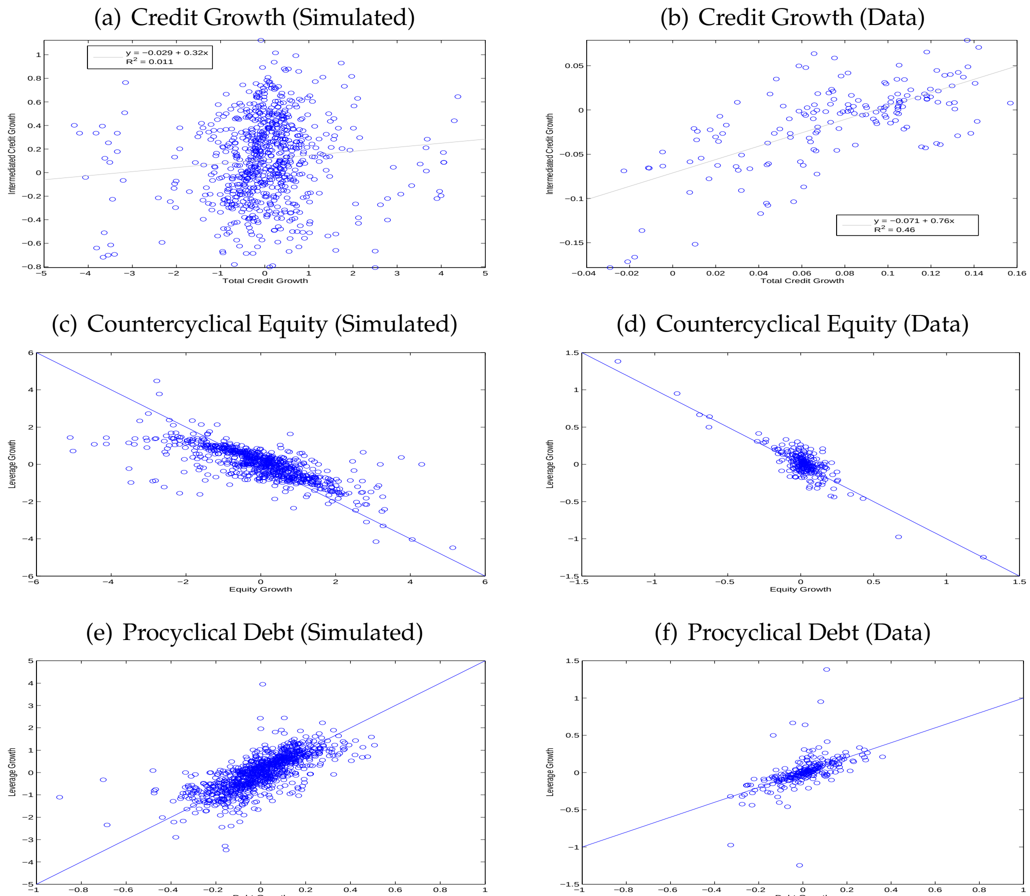
In Appendix B.3, we show that the equilibrium pricing kernel in the economy can be rewritten in terms of two observable shocks: innovations to the growth rate of intermediary leverage and innovations to output. In particular, define the standardized innovation to (log) output as

$$d\hat{y}_t = \sigma_a^{-1} \left(d \log Y_t - \mathbb{E}_t \left[d \log Y_t \right] \right) = dZ_{at},$$

and the standardized innovation to the growth rate of leverage of the intermediaries as

$$\begin{aligned} d\hat{\theta}_t &= \left(\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2 \right)^{-\frac{1}{2}} \left(\frac{d\theta_t}{\theta_t} - \mathbb{E}_t \left[\frac{d\theta_t}{\theta_t} \right] \right) \\ &= \frac{\sigma_{\theta a,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}} dZ_{at} + \frac{\sigma_{\theta\xi,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}} dZ_{\xi t}. \end{aligned}$$

Figure 5. Intermediary Balance Sheet Evolution



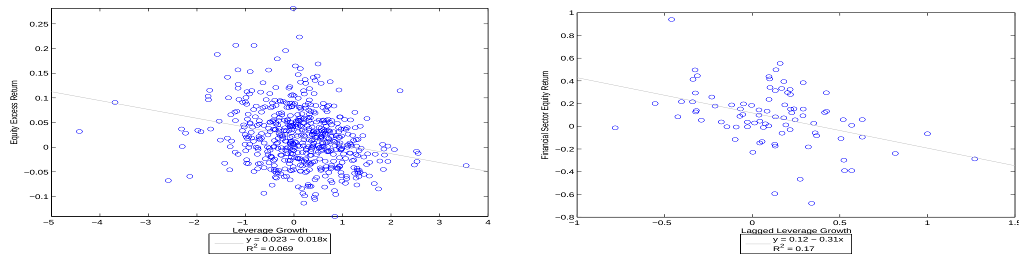
NOTES: Procyclicality of intermediary balance sheets. Top panels: The relationship between total credit in the economy and the amount of credit extended through the financial intermediary sector, with the left panel plotting the realized growth rate of capital held by intermediaries, k_t, (y-axis) versus the growth rate of total capital in the economy, K_t, (x-axis) for a representative path, and the right panel plotting the growth rate of credit extended by financial intermediaries to the non-financial corporate sector (y-axis) versus the growth rate of total credit to the non-financial corporate sector (x-axis). Middle panels: The relationship between intermediary leverage growth and intermediary equity growth, with the left panel plotting quarterly growth of intermediary leverage, θ_t, (y-axis) versus quarterly growth of intermediary wealth in the economy, ω_t, (x-axis) for a representative path, and the right panel plotting quarterly growth of broker-dealer leverage (y-axis) versus quarterly growth of scaled broker-dealer equity (x-axis). Lower panels: The relationship between intermediary leverage growth and debt growth, with the left panel plotting quarterly growth of intermediary leverage, θ_t, (y-axis) versus quarterly growth of household wealth in the economy, $1 - \omega_t$, (x-axis) for a representative path, and the right panel plotting quarterly growth of broker-dealer leverage (y-axis) versus quarterly growth of scaled broker-dealer debt (x-axis). In both the middle and the lower panels, the scaling factor is the total credit to the non-financial sector, from Flow of Funds Table L.102. Data on total credit to the nonfinancial corporate sector and the share of intermediated finance are from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets are from Flow of Funds Table L.129. Data from the model is simulated using parameters in Table 1 at a monthly frequency for 80 years.

We can express the pricing kernel as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_{ft}dt - \eta_{\theta t}d\hat{\theta}_t - \eta_{yt}d\hat{y}_t,$$

Figure 6. Excess Returns and Intermediary Leverage

(a) Simulated (b) Data



NOTES: The relationship between the growth rate of leverage of financial institutions and the equity excess returns. Right panel: quarterly excess return to holding the S&P Financial Index (y-axis) versus lagged annual growth of broker-dealer leverage (x-axis) ; left panel: quarterly excess return to holding capital, dR_{kt}, (y-axis) versus lagged annual intermediary leverage growth, $d\theta_t$, (x-axis). Data on broker-dealer leverage are from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics. Data from the model is simulated using parameters in Table 1 at a monthly frequency for 80 years.

where $\eta_{\theta t}$ and η_{yt} are equilibrium prices of risk associated with innovations to the growth rate on intermediary leverage and output, respectively. Thus pricing is similar to a two-factor Merton (1973) ICAPM, with shocks to intermediary leverage driving the uncertainty about future investment opportunities. Note, however, that the two factor structure arises in our setting not due to intertemporal hedging demands, but rather because households hedge liquidity shocks. Since capital has a negative exposure to the households' preference shocks, the price of risk associated with shocks to intermediary leverage is positive, so leverage risk commands a positive risk premium. While the sign of the risk premium is always positive, the dependence of the price of leverage risk on the leverage growth rate is nonmonotonic. The empirical literature strongly favors the positive price of leverage risk for stock and bond returns (see Adrian, Etula, and Muir, 2014) and a negative relationship between the price of risk and the growth rate of leverage (see Adrian, Moench, and Shin, 2010, 2014).

The left panel of Figure 6 plots simulated excess returns as a function of intermediary leverage growth, while the right panel plots the same relationship in the data. In particular, we see that the excess return to capital increases as the growth rate of intermediary leverage decreases. This negative relationship within the model is further documented in Table 4, with the linear regression coefficient consistently negative across different path realizations.

Unlike the price of leverage risk, the price of risk associated with shocks to output changes signs, depending on whether the equilibrium sensitivity of the return to holding capital to output shocks is lower or higher than the fundamental volatility. The time-varying nature of the direction of the

	Data	Mean	5%	Median	95%
β_0	0.118	0.076	0.068	0.076	0.084
β_1	-0.310	-0.031	-0.038	-0.031	-0.024
R^2	0.167	0.100	0.064	0.100	0.143

Table 4: Excess Returns and Intermediary Leverage

NOTES: The relationship between excess returns and lagged broker-dealer leverage growth. The "Data" column reports the coefficients estimated using the quarterly return to holding the S&P Financial Index as the dependent variable, and lagged annual broker-dealer leverage growth as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized quarterly excess return to holding capital, dR_{kt} as the dependent variable, and lagged annual intermediary leverage growth, $d\theta_t$, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on broker-dealer leverage are from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics and Barclays.

risk premium for output shocks makes it difficult to detect in observed returns, suggesting an explanation for the poor empirical performance of the production CAPM.

4 Financial Stability and Household Welfare

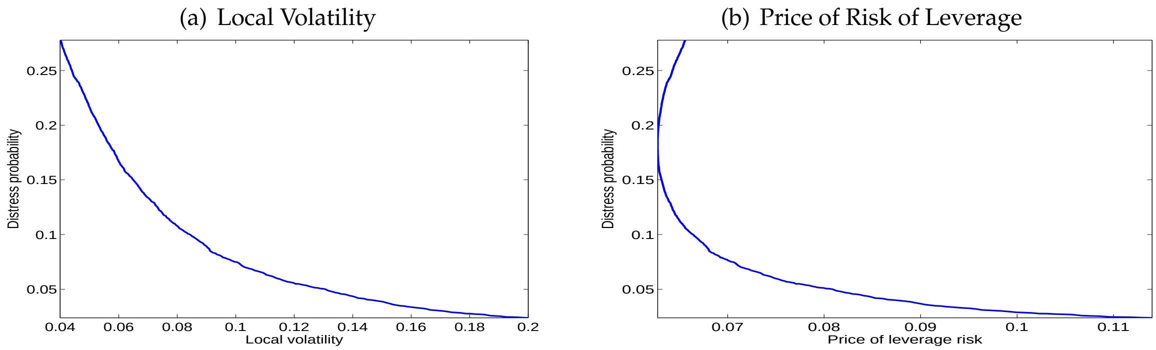
In this Section, we describe the term structure of the distress probability, $\delta_t(T)$, and, in particular, the effect of a tightening of the risk-based capital constraint. We then compare the equilibrium outcomes in our model to the equilibrium outcomes in one with constant leverage. Finally, we discuss some implications of the risk-based capital constraint for the welfare of the households in the economy.

4.1 Intermediary distress

We begin by considering the trade-off between the instantaneous riskiness of capital investment and the long-run fragilities in the economy. The left panel of Figure 7 plots the six month distress probability[9] as a function of the current instantaneous volatility of the return to holding capital. We see that the model-implied quantities have the negative relationship observed in the run-up to the 2007-2009 financial crisis. This relation forms the crux of the volatility paradox: Periods of low volatility of the return to holding capital coincide with high intermediary leverage, which leads to high systemic solvency and liquidity risk. The volatility paradox was first described by Brunnermeier and Sannikov (2014), and empirically documented by Adrian and Brunnermeier

[9] Although this probability cannot be computed analytically, we can easily compute it using Monte Carlo simulations.

Figure 7. Volatility Paradox



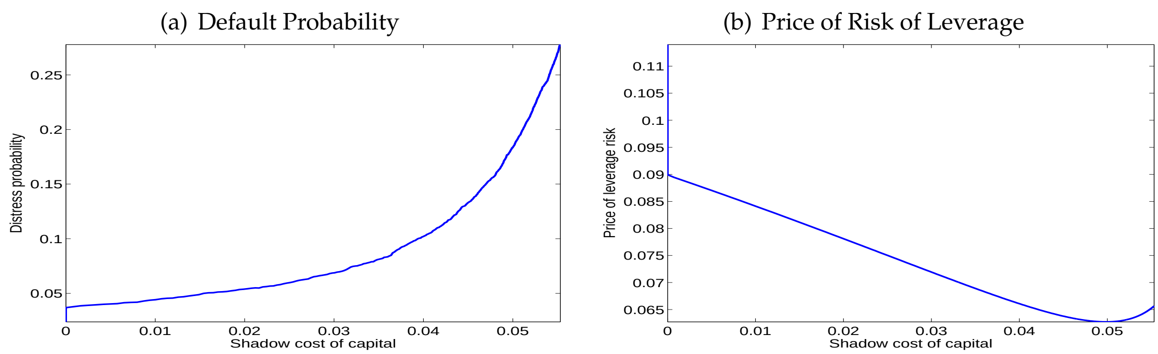
NOTES: Left panel: 6 month probability of intermediary default (y-axis) versus instantaneous volatility of equity returns, $\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$, ($x$-axis); right panel: 6 month probability of intermediary default (y-axis) versus the risk price of standardized shocks to leverage, $\eta_{\theta t}$, (x-axis). The default probabilities are computed using 10000 simulations of the economy on a monthly frequency using the parameters in Table 1.

(2011). In the context of the model, local volatility is inversely proportional to leverage. As leverage increases, the intermediaries issue more risky debt, making distress more likely. This leads to the negative relationship between the probability of distress and current period return volatility. The right panel of Figure 7 plots the trade-off between the six month distress probability and the price of risk associated with shocks to the growth rate of intermediary leverage. Since the price of leverage risk depends linearly on return volatility, an increase in contemporaneous risk increases the price of leverage risk while decreasing the long-term instability in the economy. This mechanism allows intermediaries to increase their risk exposure during periods of low volatility, which increases the risk of financial distress.

In Figure 8, we plot the trade-off between the shadow cost of capital, ζ_t, faced by the intermediaries and the risk in the economy. As the price of leverage risk increases, it becomes more costly for intermediaries to increase their leverage, increasing their shadow cost of capital (right panel). In the presence of the systemic risk-return trade-off, this implies that the shadow cost of capital increases as the probability of distress decreases (left panel). Intuitively, the shadow cost of increasing leverage is highest when the intermediary is safest: An extra unit of leverage has a marginally higher impact on the probability of distress for intermediaries with low leverage.

Intermediary distress is costly (in consumption terms) for the households. In Figure 9, we plot a sample evolution of the economy, focusing on the evolution of consumption (upper panel), intermediary wealth share in the economy and intermediary leverage (middle panels), and of the

Figure 8. Shadow Cost of Capital



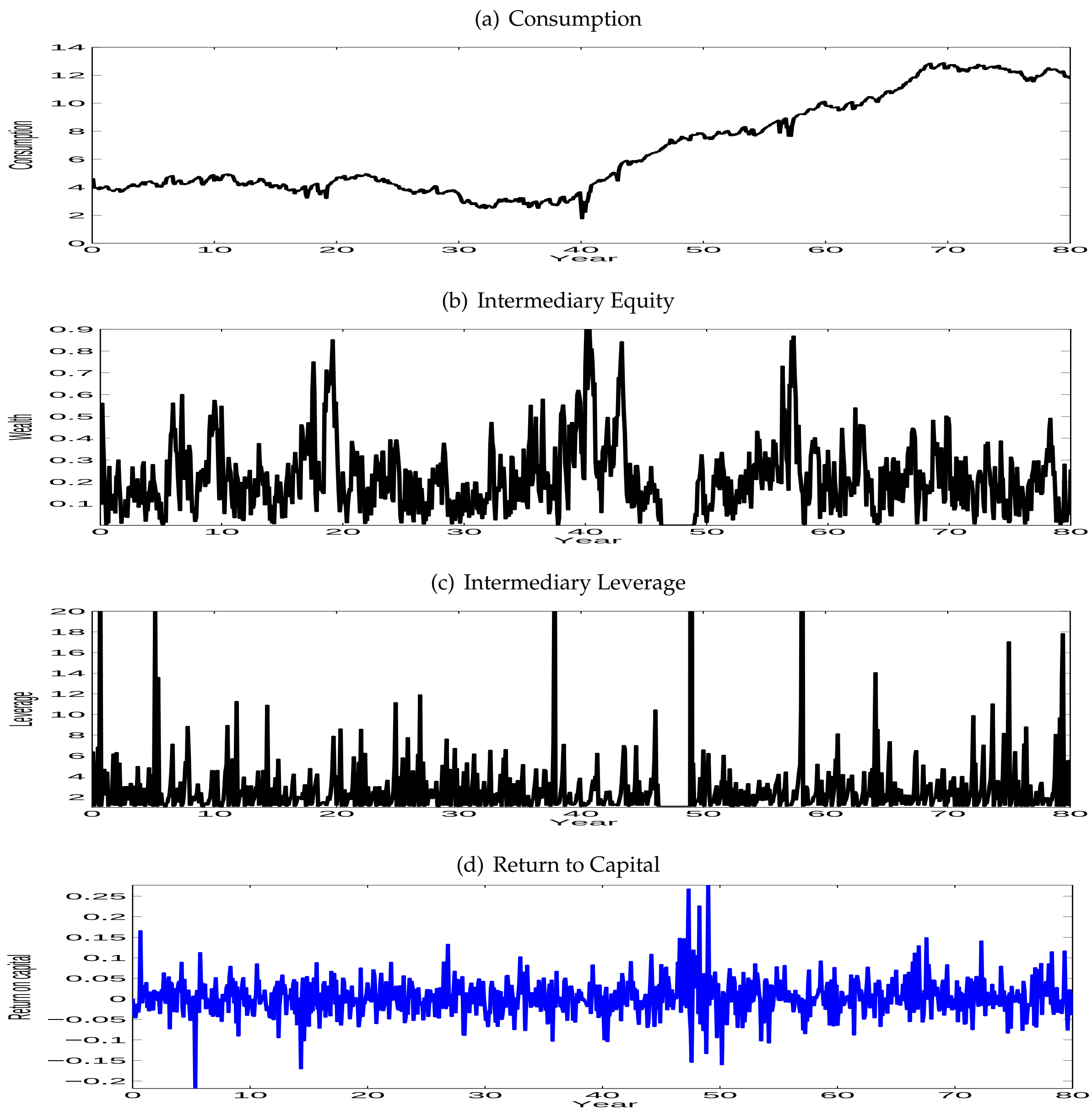
NOTES: Left panel: 6 month probability of intermediary default (y-axis) versus the shadow cost of increased leverage, ζ_t, (x-axis); right panel: the risk price of standardized shocks to leverage, $\eta_{\theta t}$, (y-axis) versus the shadow cost of increased leverage, ζ_t, (x-axis). The default probabilities are computed using 10000 simulations of the economy on a monthly frequency using the parameters in Table 1.

realized return to intermediary debt (lower panel). Notice first that, while intermediaries' distress is usually preceded by high intermediary leverage, distress can occur even when intermediary leverage is relatively low. Moreover, intermediaries can maintain high levels of leverage without becoming distressed. Thus, high leverage is not a foolproof indicator of distress risk. The recapitalization of intermediaries comes at the cost of a consumption drop for the households, which can be quite significant. Since the restructuring of intermediaries is done through default on debt, household wealth (and, hence, consumption) exhibits sharp declines when intermediaries become distressed. It is worth emphasizing that the transmission mechanism from financial sector distress to real economic activity is via two channels. The first is a wealth effect of households, which leads to an adjustment of the consumption path, and a reallocation of savings. The second channel is more direct, and consists in adjustments to the capital creation decision of intermediaries.

4.2 Distortions and amplifications

The simulated path of the economy in Figure 9 illustrates the negative implications of intermediary distress for the households in the economy. The risk-based capital constraint faced by the intermediaries in our economy amplifies the fundamental shocks and distorts equilibrium outcomes. An adverse shock to the relative wealth of the intermediaries reduces the equilibrium level of investment and leads to a lower price of capital, which makes the risk-based capital constraint bind more, reducing further financial intermediaries' effective risk taking. The amplification mecha-

Figure 9. Sample Path of the Economy



nism acts through the time-varying leverage constraint that is induced by the risk-sensitive capital constraint.

To understand the mechanism better, we describe the equilibrium outcomes in an economy with constant leverage, and contrast the resulting dynamics with those in the full model. In particular, consider an economy in which, instead of facing the risk-based capital constraint, the intermediaries face a constant leverage constraint, such that

$$\frac{p_{kt}A_t k_t}{w_t} = \bar{\theta},$$

where $\bar{\theta}$ is a constant set by the prudential regulator. The equilibrium outcomes are summarized in the following lemma.

Lemma 4.1. *The economy with constant leverage converges to an economy with a constant wealth share of the intermediary sector in the economy*

$$\omega_t = \bar{\theta}^{-1}.$$

In the steady state, the intermediary sector owns all the capital in the economy, with the expected excess return to holding capital given by

$$\mu_{Rk,t} - r_{ft} = \frac{1}{p_k} + \sigma_a^2 - \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \Phi\left(i_t\right),$$

and the expected excess return to holding bank debt given by

$$\mu_{Rb,t} - r_{ft} = \sigma_a^2,$$

with the riskiness of the returns equal to the riskiness of the productivity growth

$$\sigma_{ka,t} = \sigma_{ba,t} = \sigma_a$$
$$\sigma_{k\xi,t} = \sigma_{b\xi,t} = 0.$$

Proof. See Appendix C. □

Thus, when the financial intermediaries face a constant leverage constraint, the intermediary sector does not amplify the fundamental shocks in the economy. Furthermore, since intermediaries represent a constant fraction of the wealth of the economy with constant leverage, there is no risk of intermediary distress. Notice, however, that the excess return to holding capital compensates investors for the cost of capital adjustment. Thus, the financial system provides a channel through which market participants can share the cost of capital investment. Importantly, the household preference shock ξ is not transmitted in this economy. Intuitively, since the households aren't the marginal investors in the capital market, the price of capital only reflects shocks to intermediaries' pricing kernel which only varies with productivity shocks.

The benefit of having a financial system with a flexible leverage constraint is, then, increased output growth and more valuable capital, albeit at the cost of financial and economic stability. Since the rate of investment and the capital price are constant in this benchmark, the volatility of consumption growth equals the volatility of productivity growth, and the expected consumption growth rate equals the expected productivity growth rate. In our model, the financial intermediary sector allows households to smooth consumption, reducing the instantaneous volatility of consumption during good times, but at the cost of higher consumption growth volatility during times of financial distress. In particular, notice that, in the model with risk-based capital constraints, volatility of consumption growth is given by

$$\left\langle \frac{dc_t}{c_t} \right\rangle^2 = \left(-\frac{2\theta_t \omega_t}{\beta(1-\omega_t)} p_{kt} \left(\sigma_{ka,t} - \sigma_a \right) + \sigma_a \right)^2 + \left(\frac{2\theta_t \omega_t}{\beta(1-\omega_t)} p_{kt} \sigma_{k\xi,t} \right)^2,$$

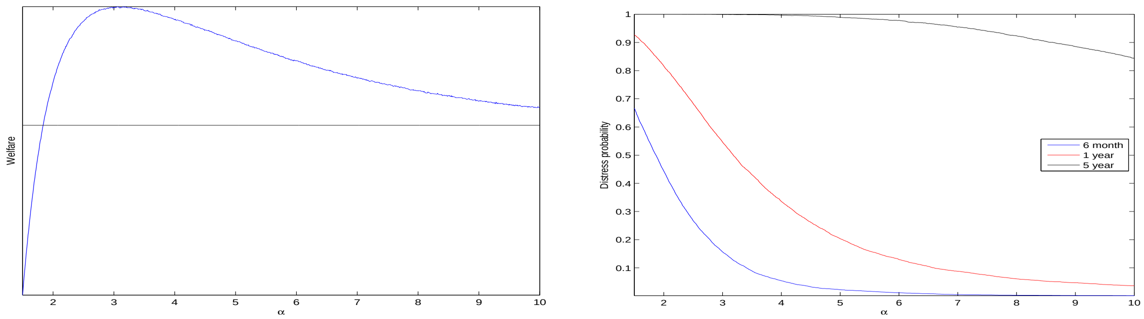
which is lower than the fundamental volatility σ_a^2 when $\sigma_{ka,t}$ is bigger than σ_a.

More formally, consider the trade-off in terms of the expected discounted present value of household utility. In Figure 10, we plot the household welfare in the economy with pro-cyclical intermediary leverage as a function of the tightness of the risk-based capital constraint, as well as the household welfare in the economy with constant leverage. Notice first that household welfare is not monotone in α: Initially, as the risk-based capital constraint becomes tighter, household welfare increases as distress risk decreases. For high enough levels of α, however, the household welfare decreases as the risk-based capital constraint becomes tighter. Intuitively, for low values of α, periods of financial distress (which are accompanied by sharp drops in consumption) are more frequent and the households become better off as the constraint becomes tighter. As α increases, the intermediaries become more stable, increasing household welfare. As α becomes too large, while probability of intermediary distress is still lower (see the right panel of Figure 10), the risk-sharing function of the intermediaries is impeded, leading to lower household utility. Notice finally that household welfare in the economy with pro-cyclical leverage can be higher than that in the economy with constant leverage, even when a suboptimal α is chosen.

Note, however, that the risk based capital constraint does not necessarily constitute optimal policy in our setup. Instead, we view the risk based capital constraint as being imposed by regulators in order to solve moral hazard and adverse selection problems that we do not model explicitly.

Figure 10. Household Welfare

(a) Household Welfare (b) Probability of Distress



NOTES: Left panel: expected present value of household utility (y-axis) as a function of the tightness of risk-based capital constraint, α, (x-axis); right panel: 6 month, 1 year and 5 year cumulative default probabilities (y-axis) as a function of the tightness of risk-based capital constraint, α, (x-axis). Welfare and default probabilities are computed using 10000 simulations of the economy on a monthly frequency using the parameters in Table 1, with household welfare computed over a 70 year horizon.

Within our setting, welfare could be improved if intermediaries were allowed to issue equity to households. In practice, such adjustments are likely costly, thus implying a non-trivial equity issuance decision by intermediaries. We leave the study of such a setting to future research.

4.3 Stress tests

By introducing preferences for the financial intermediaries, we can extend our model to study the impact of the use of stress tests as a macroprudential tool. By further introducing preferences for the prudential regulator, the model also provides implications for the optimal design of stress tests. We leave the formal treatment of these extensions for future work and provide here a sketch of how stress tests can be incorporated in the current setting.

Recall that, in our model, intermediary debt is subject to the risk-based capital constraint, which is a constraint on the local volatility of the asset side of the intermediary balance sheet

$$\theta_t^{-1} \geq \alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}.$$

Stress tests, on the other hand, can be interpreted as a constraint on the total volatility of the asset side of the balance sheet over a fixed time interval

$$\theta_t^{-1} \geq \vartheta \sqrt{\mathbb{E}_t \left[\int_t^T \left(\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2 \right) ds \right]}.$$

Thus, in effect, stress tests can be thought of as a Stackelberg game between the policymaker and the financial intermediaries, with the policymaker moving first to choose the maximal allowable level of volatility over a time interval, and the intermediaries moving second to allocate the volatility allowance between different periods. Under the assumption that the prudential regulator designs stress tests to minimize total volatility, while the intermediaries maximize the expected discounted value of equity, the optimization problem for the intermediaries resembles the optimal robust control problem under model misspecification studied by Hansen, Sargent, Turmuhambetova, and Williams (2006); Hansen and Sargent (2001); Hansen, Sargent, and Tallarini (1999); Hansen and Sargent (2007), among others

$$V_t(\vartheta) = \max_{\{i,\beta,k\}} \min_{q \in \mathcal{Q}(\vartheta)} \int \int_t^{\tau_D} e^{-\rho(s-t)} w_t(i,\beta,k)\, ds dq$$

subject to

$$\theta_t^{-1} \geq \vartheta \sqrt{\int_t^T \int \left(\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2\right) dq_s ds}.$$

Notice that, in the limit at $T \to t + dt$, this reduces to the risk-based capital constraint described above. In the language of Hansen, Sargent, Turmuhambetova, and Williams (2006), this is a *nonsequential* problem since the constraint is over a non-infinitesimal time horizon. The density function q is a density over the future realizations of the fundamental shocks $\left(dZ_{at}, dZ_{\xi t}\right)$ in the economy, and $\mathcal{Q}$ is the set of densities that satisfies the stress-test constraint. Hansen, Sargent, Turmuhambetova, and Williams (2006) show how to move from the nonsequential robust controls problems to sequential problems. In particular, for the constraint formulation, they augment the state-space to include the continuation value of entropy and solve for the optimal value function that also depends on this continuation entropy.

In our setting, we can reformulate the optimization problem of the representative intermediary as

$$V_t(\vartheta) = \max_{\{i,\beta,k,\alpha_s\}} \mathbb{E}_t \left[\int_t^{\tau_D} e^{-\rho(s-t)} w_t(i,\beta,k)\, ds\right]$$

subject to

$$\frac{\theta_s^{-1}}{\alpha_s} \geq \sqrt{\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2}$$

$$\theta_t^{-1} \geq \vartheta \sqrt{\mathbb{E}_t \left[\int_t^T \frac{\theta_s^{-2}}{\alpha_s^2} ds \right]}.$$

That is, the intermediaries choose an optimal capital plan at the time of the stress test to maximize the discounted present value of equity subject to satisfying the intertemporal volatility constraint imposed by the stress test. Locally, the portfolio allocation decision of the intermediaries satisfy a risk-based capital constraint, albeit with a time-varying α. However, along a given capital plan, the optimal decisions of both the households and the intermediaries are as described above. Stress tests are hence a natural but technically challenging extension of the current setup and are left for future exploration.

5 Conclusion

We present a dynamic, general equilibrium theory of financial intermediaries' leverage cycle as a conceptual basis for policies geared toward financial stability. In this setup, any change in prudential policies has general equilibrium effects that impact the pricing of financial and nonfinancial credit, the equilibrium volatilities of financial and real assets, and the allocation of consumption and investment goods. From a normative point of view, such effects are important to understand, as they ultimately determine the effectiveness of prudential policies.

The assumptions of our model are empirically motivated, and our theory captures many important stylized facts about financial intermediary dynamics that have been documented in the literature. There is both direct and intermediated credit by households, giving rise to substitution from intermediated credit to directly granted credit in times of tighter intermediary constraints. The risk-based funding constraint leads to procyclical intermediary leverage, matching empirical observations. Our theory generates the volatility paradox: times of low contemporaneous volatility allow high intermediary leverage, increases in forward-looking systemic risk. Finally, the time variation in the pricing of risk is a function of leverage growth and the price of risk of asset exposure is positive, two additional features that are strongly borne out in the data.

The most important contribution of the paper is to directly study the impact of prudential policies on the likelihood of systemic risk. We uncover a systemic risk-return trade-off: Tighter intermediary capital requirements tend to shift the term structure of systemic risk downward, at the cost of increased risk pricing today. This trade-off forms the basis for the evaluation of costs and benefits associated with financial stability policies.

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A Proofs

A.1 Household's optimization

Recall that the household solves the portfolio optimization problem:

$$\max_{\{c_t,\pi_{kt},\pi_{bt}\}} \quad \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

subject to the wealth evolution equation:

$$\begin{aligned} dw_{ht} &= r_{ft}w_{ht}dt + w_{ht}\pi_{kt}\left\{\left(\mu_{Rk,t}-r_{ft}\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t}\right\} \\ &\quad + w_{ht}\pi_{bt}\left\{\left(\mu_{Rb,t}-r_{ft}\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t}\right\} - c_t dt, \end{aligned}$$

and the no-shorting constraints:

$$\pi_{kt},\ \pi_{bt} \geq 0.$$

Instead of solving the dynamic optimization problem, we follow Cvitanić and Karatzas (1992) and rewrite the household problem in terms of a static optimization. Cvitanić and Karatzas (1992) extend the Cox and Huang (1989) martingale method approach to constrained optimization problems, such as the one that the households face in our economy.
Define $K = \mathbb{R}_+^2$ to be the convex set of admissible portfolio strategies and introduce the support function of the set $-K$ to be

$$\begin{aligned} \delta(x) = \delta(x|K) &\equiv \sup_{\vec{\pi}\in K}\left(-\vec{\pi}'x\right) \\ &= \begin{cases} 0, & x\in K \\ +\infty, & x\notin K \end{cases}. \end{aligned}$$

We can then define an auxiliary unconstrained optimization problem for the household, with the returns in the auxiliary asset market defined as

$$\begin{aligned} r_{ft}^v &= r_{ft} + \delta(\vec{v}_t) \\ dR_{kt}^v &= \left(\mu_{Rk,t} + v_{1t} + \delta(\vec{v}_t)\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t} \\ dR_{bt}^v &= \left(\mu_{Rb,t} + v_{2t} + \delta(\vec{v}_t)\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t}, \end{aligned}$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V(K)$ of square-integrable, progressively measurable processes taking values in K. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta(\vec{v}_t)\right)dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma_{Rt}'\right)^{-1}d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative household then becomes

$$\max_{c_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right]$$

subject to the static budget constraint:

$$w_{h0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_t dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves the dual problem

$$\min_{v \in V(K)} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t} \tilde{u}\left(\lambda \eta_t^v\right) dt\right],$$

where $\tilde{u}(x)$ is the convex conjugate of $-u(-x)$

$$\tilde{u}(x) \equiv \sup_{z>0} \left[\log(zx) - zx\right] = -\left(1 + \log x\right)$$

and λ is the Lagrange multiplier of the static budget constraint. Cvitanić and Karatzas (1992) show that, for the case of logarithmic utility, the optimal choice of v satisfies

$$\begin{aligned} v_t^* &= \arg\min_{x \in K} \left\{2\delta(x) + \left|\left|\left(\vec{\mu}_{Rt} - r_{ft} + x\right)' \sigma_{Rt}^{-1}\right|\right|^2\right\} \\ &= \arg\min_{x \in K} \left|\left|\left(\vec{\mu}_{Rt} - r_{ft} + x\right)' \sigma_{Rt}^{-1}\right|\right|^2. \end{aligned}$$

Thus,

$$\begin{aligned} v_{1t} &= \begin{cases} 0, & \mu_{Rk,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rk,t}, & \mu_{Rk,t} - r_{ft} < 0 \end{cases} \\ v_{2t} &= \begin{cases} 0, & \mu_{Rb,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rb,t}, & \mu_{Rb,t} - r_{ft} < 0 \end{cases}. \end{aligned}$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\xi_t - \rho_h t}}{c_t} - \lambda \eta_t^v,$$

or

$$c_t = \frac{e^{-\xi_t - \rho_h t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ht} = \mathbb{E}_t\left[\int_t^{+\infty} \eta_s^v c_s ds\right] = \mathbb{E}_t\left[\int_t^{+\infty} \frac{e^{-\xi_s - \rho_h s}}{\lambda} ds\right] = \frac{e^{-\xi_t - \rho_h t}}{\lambda\left(\rho_h - \sigma_\xi^2/2\right)}.$$

Thus

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

To solve for the household's optimal portfolio allocation, notice that:

$$\begin{aligned}\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} &= -\rho_h dt - d\xi_t + \frac{1}{2} d\xi_t^2 \\ &= \left(-\rho_h + \frac{1}{2}\sigma_\xi^2\right) dt - \sigma_\xi \rho_{\xi,a} dZ_{at} - \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} dZ_{\xi t}.\end{aligned}$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ht}}{w_{ht}} + \frac{dw_{ht}}{w_{ht}} \frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\pi}_t' = \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)' \left(\sigma_{Rt}'\sigma_{Rt}\right)^{-1} - \sigma_\xi \begin{bmatrix}\rho_{\xi a} & \sqrt{1-\rho_{\xi a}^2}\end{bmatrix} \sigma_{Rt}^{-1}.$$

A.2 Intermediary optimization

Recall that the representative intermediary solves

$$\max_{\theta_t, i_t} \quad \mathbb{E}_t\left[\frac{dw_t}{w_t}\right] - \frac{\gamma}{2}\mathbb{V}_t\left[\frac{dw_t}{w_t}\right],$$

subject to the dynamic intermediary budget constraint

$$\frac{dw_t}{w_t} = \theta_t\left(dR_{kt} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right) dt - r_{ft}dt\right) - \left(\theta_t - 1\right)\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt,$$

and the risk-based capital constraint constraint

$$\theta_t \leq \alpha^{-1}\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right)^{-\frac{1}{2}}.$$

Forming the Lagrangian, we obtain

$$\begin{aligned}\mathcal{L}_t = \max_{\theta_t, i_t} \quad & \theta_t\left(\mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right) - r_{ft}\right) - \left(\theta_t - 1\right)\left(\mu_{Rb,t} - r_{ft}\right) \\ & - \frac{\gamma}{2}\left[\left(\theta_t\sigma_{ka,t} - \left(\theta_t - 1\right)\sigma_{ba,t}\right)^2 + \left(\theta_t\sigma_{k\xi,t} - \left(\theta_t - 1\right)\sigma_{b\xi,t}\right)^2\right] \\ & + \zeta_t\left(\alpha^{-1}\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right)^{-\frac{1}{2}} - \theta_t\right),\end{aligned}$$

where ζ_t is the Lagrange multiplier on the risk-based capital constraint. Taking the first order conditions, we obtain

$$\begin{aligned}[i_t]: \quad & 0 = \Phi'\left(i_t\right) - p_{kt}^{-1} \\ [\theta_t]: \quad & 0 = \left(\mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right) - r_{ft}\right) - \left(\mu_{Rb,t} - r_{ft}\right) - \zeta_t \\ & - \gamma\left(\sigma_{ka,t} - \sigma_{ba,t}\right)\left(\theta_t\left(\sigma_{ka,t} - \sigma_{ba,t}\right) - \sigma_{ba,t}\right) - \gamma\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\left(\theta_t\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right) - \sigma_{b\xi,t}\right).\end{aligned}$$

Consider first the case when the intermediary is unconstrained in his leverage choice, so that $\zeta_t = 0$. Then, solving for the optimal leverage choice, we obtain

$$\theta_t = \frac{\left(\mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right) - r_{ft}\right) - \left(\mu_{Rb,t} - r_{ft}\right)}{\gamma\left[\left(\sigma_{ka,t} - \sigma_{ba,t}\right)^2 + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)^2\right]} + \frac{\sigma_{ba,t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right) + \sigma_{b\xi,t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)}{\left[\left(\sigma_{ka,t} - \sigma_{ba,t}\right)^2 + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)^2\right]}.$$

Consider now the case when the intermediary is constrained. Solving for the Lagrange multiplier, we obtain

$$\zeta_t = \left(\mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right) - r_{ft}\right) - \left(\mu_{Rb,t} - r_{ft}\right) + \gamma\left[\sigma_{ba,t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right) + \sigma_{b\xi,t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\right]$$
$$- \gamma\theta_t\left[\left(\sigma_{ka,t} - \sigma_{ba,t}\right)^2 + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)^2\right].$$

B Equilibrium outcomes

In this Appendix, we provide the details of the derivation of the equilibrium outcomes.

B.1 Capital evolution

Recall from the intermediary's leverage constraint that

$$\theta_t = \frac{p_{kt} A_t k_t}{w_t}.$$

Using our definition of ω_t, we can thus express the amount of capital held by the financial institutions as

$$k_t = \frac{\theta_t w_t}{p_{kt} A_t} = \theta_t \omega_t K_t.$$

Applying Itô's lemma, we obtain

$$dk_t = \omega_t K_t d\theta_t + \theta_t K_t d\omega_t + \theta_t \omega_t dK_t + K_t \left\langle d\theta_t, d\omega_t \right\rangle.$$

Recall that the intermediary's capital evolves as

$$dk_t = \left(\Phi\left(i_t\right) - \lambda_k\right) k_t dt.$$

Equating coefficients, we obtain

$$\begin{aligned}
\sigma_{\theta a,t} &= -\sigma_{\omega a,t} \\
\sigma_{\theta \xi,t} &= -\sigma_{\omega \xi,t} \\
\mu_{\theta t} &= \underbrace{\Phi\left(i_t\right)\left(1 - \theta_t \omega_t\right)}_{\text{asset growth rate}} - \mu_{\omega t} + \underbrace{\sigma^2_{\theta a,t} + \sigma^2_{\theta \xi,t}}_{\text{risk adjustment}}.
\end{aligned}$$

Thus, intermediary leverage is perfectly negatively correlated with the share of wealth held by the financial intermediaries. This reflects the fact that capital stock is not immediately adjustable, so changes in the value of intermediary assets translate one-for-one into changes in intermediary leverage. Notice further that the intermediary faces a trade-off in the growth rate of its leverage, $\mu_{\theta t}$, and the growth rate of its wealth share in the economy, $\mu_{\omega t}$.

B.2 Intermediary wealth evolution

Turn now to the equilibrium evolution of intermediaries' wealth. Recall that we have defined the fraction of total wealth in the economy held by the intermediaries as

$$\omega_t = \frac{w_t}{p_{kt} A_t K_t}.$$

Applying Ito's lemma, we obtain

$$\frac{d\omega_t}{\omega_t} = \frac{dw_t}{w_t} - \frac{d\left(p_{kt}A_t\right)}{p_{kt}A_t} - \frac{dK_t}{K_t} + \left\langle \frac{d\left(p_{kt}A_t\right)}{p_{kt}A_t} \right\rangle^2 - \left\langle \frac{dw_t}{w_t}, \frac{d\left(p_{kt}A_t\right)}{p_{kt}A_t} \right\rangle.$$

Recall further that

$$\begin{aligned}\frac{dw_t}{w_t} &= \theta_t\left(dr_{kt} - r_{ft}dt\right) - \left(\theta_t - 1\right)\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt \\ &= \theta_t\left[\left(\mu_{Rk,t} - r_{ft} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t}\right] \\ &\quad - \left(\theta_t - 1\right)\left[\left(\mu_{Rb,t} - r_{ft}\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{d\left(p_{kt}A_t\right)}{p_{kt}A_t} &= \left(\mu_{Rk,t} + \lambda_k - \frac{1}{p_{kt}}\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t} \\ \frac{dK_t}{K_t} &= \left(\Phi\left(i_t\right)\theta_t\omega_t - \lambda_k\right)dt.\end{aligned}$$

Thus, the expected rate of change in the financial intermediaries' wealth share in the economy is given by

$$\begin{aligned}\mu_{\omega t} = &\underbrace{\left(\theta_t - 1\right)\left(\mu_{RKt} - \mu_{Rb,t}\right)}_{\text{expected porfolio return}} - \underbrace{\left(\sigma_{ka,t}\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right)}_{\text{compensation for portfolio risk}} \\ &+ \underbrace{\frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right]}_{\text{consumption provision to households}},\end{aligned}$$

where we have used the goods market clearing condition and the households' optimal consumption rate to substitute

$$1 = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)p_{kt}\left(1 - \omega_t\right) + i_t\theta_t\omega_t.$$

The loadings of the financial intermediaries' wealth share in the economy on the two sources of fundamental risk are given by

$$\begin{aligned}\sigma_{\omega a,t} &= (\theta_t - 1)\left(\sigma_{ka,t} - \sigma_{ba,t}\right)\\ \sigma_{\omega\xi,t} &= (\theta_t - 1)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right).\end{aligned}$$

That is, the risk loadings of the financial intermediaries' relative wealth reflect the ability of the financial intermediaries to absorb shocks to their balance sheets. The negative sign on the volatility of bond returns reflects the fact that losses in the value of the bonds benefit the intermediaries by reducing their debt burden.

B.3 Equilibrium pricing kernel

Using the households' optimal portfolio choice, we can express the pricing kernel in terms of exposures to the fundamental shocks $\left(dZ_{at}, dZ_{\xi t}\right)$ as

$$\begin{aligned}\frac{d\Lambda_t}{\Lambda_t} = -r_{ft}dt &- \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{ka,t} + \frac{\omega_t\left(\theta_t-1\right)}{1-\omega_t}\sigma_{ba,t} + \sigma_\xi\rho_{\xi,a}\right)dZ_{at}\\ &- \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{k\xi,t} + \frac{\omega_t\left(\theta_t-1\right)}{1-\omega_t}\sigma_{b\xi,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\right)dZ_{\xi t}.\end{aligned}$$

While it is natural to express the pricing kernel as a function of the fundamental shocks ξ and a, these are not readily observable. Instead, we follow the empirical literature and express the pricing kernel in terms of shocks to output and leverage. Define the standardized innovation to (log) output as

$$d\hat{y}_t = \sigma_a^{-1}\left(d\log Y_t - \mathbb{E}_t\left[d\log Y_t\right]\right) = dZ_{at},$$

and the standardized innovation to the growth rate of leverage of the intermediaries as

$$\begin{aligned}d\hat{\theta}_t &= \left(\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2\right)^{-\frac{1}{2}}\left(\frac{d\theta_t}{\theta_t} - \mathbb{E}_t\left[\frac{d\theta_t}{\theta_t}\right]\right)\\ &= \frac{\sigma_{\theta a,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}}dZ_{at} + \frac{\sigma_{\theta\xi,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}}dZ_{\xi t}.\end{aligned}$$

Thus, we can express the pricing kernel as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_{ft}dt - \eta_{\theta t}d\hat{\theta}_t - \eta_{yt}d\hat{y}_t,$$

where the price of risk associated with shocks to the growth rate of intermediary leverage is

$$\eta_{\theta t} = \sqrt{1 + \frac{\left(\sigma_{ka,t} - \sigma_a\right)^2}{\sigma_{k\xi,t}^2}}\left(-\frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_{k\xi,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\right),$$

and the price of risk associated with shocks to output is

$$\eta_{yt} = \sigma_a + \sigma_\xi\left(\rho_{\xi,a} - \frac{\sigma_{ka,t} - \sigma_a}{\sigma_{k\xi,t}}\sqrt{1-\rho_{\xi,a}^2}\right).$$

B.4 Equilibrium capital price

Recall that goods market clearing implies the households consume all output, except that used for investment

$$c_t = A_t \left(K_t - i_t k_t \right).$$

Substituting the optimal investment choice of the intermediary, we can express the goods market clearing condition as

$$\underbrace{\left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t \right)}_{\text{household demand}} = \underbrace{1 - \frac{\theta_t \omega_t}{\phi_1} \left(\frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1 \right)}_{\text{total supply}}.$$

The households' demand for the consumption good is driven by the households' wealth share in the economy, $1 - \omega_t$, and the capital price p_{kt}. The supply of the consumption good, on the other hand, is determined by the financial intermediaries' wealth share in the economy, ω_t, financial intermediaries' leverage, θ_t, and the capital price. Denoting

$$\beta = \left(\frac{4}{\phi_0^2 \phi_1} \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) \right),$$

the price of capital solves

$$0 = p_{kt}^2 \theta_t \omega_t + \beta p_{kt} \left(1 - \omega_t \right) - \frac{4}{\phi_0^2 \phi_1} - \frac{4 \theta_t \omega_t}{\phi_0^2 \phi_1^2},$$

or

$$p_{kt} = \frac{-\beta \left(1 - \omega_t \right) + \sqrt{\beta^2 \left(1 - \omega_t \right)^2 + \frac{16}{\phi_0^2 \phi_1^2} \theta_t \omega_t \left(\phi_1 + \theta_t \omega_t \right)}}{2 \theta_t \omega_t}. \tag{B.1}$$

As an aside, notice that, for the intermediary to disinvest, we must have

$$\left(1 - \omega_t \right) \geq \frac{\phi_0 \phi_1}{2 \left(\rho_h - \frac{\sigma_\xi^2}{2} \right)}.$$

Thus, the intermediary disinvests when the household is a large fraction of the economy—that is, when the intermediary has a relatively low value of equity. Applying Itô's lemma and equating coefficients, we obtain

$$\begin{aligned}
[dZ_{at}] : \quad & \beta \omega_t \sigma_{\omega a,t} = \left(2 \theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t \right) \right) \left(\sigma_{ka,t} - \sigma_a \right) \\
[dZ_{\xi t}] : \quad & \beta \omega_t \sigma_{\omega \xi,t} = \left(2 \theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t \right) \right) \sigma_{k\xi,t} \\
[dt] : \quad & 0 = \left(p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \left(1 - \theta_t \omega_t \right) \right) \theta_t \omega_t \Phi \left(i_t \right) \left(1 - \theta_t \omega_t \right) \\
& + \left(2 \theta_t \omega_t p_{kt} + \beta \left(1 - \omega_t \right) \right) p_{kt} \left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a \sigma_{ka,t} \right)
\end{aligned}$$

$$
\begin{aligned}
&- \beta p_{kt}\omega_t\mu_{\omega t} + \theta_t\omega_t p_{kt}^2 \left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2 \right) \\
&- \beta p_{kt}\omega_t \left(\left(\sigma_{ka,t} - \sigma_a\right)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t} \right).
\end{aligned}
$$

Thus, in equilibrium, the financial intermediaries' wealth ratio in the economy reacts to shocks in the households' beliefs in the same direction as the return to capital.

B.5 Solution

To summarize, in equilibrium, we must have

$$
\begin{aligned}
&\mu_{\theta t} = \Phi\left(i_t\right)\left(1 - \theta_t\omega_t\right) - \mu_{\omega t} + \sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2 \\
&\mu_{Rk,t} - r_{ft} = \left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right)\frac{1-\theta_t\omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \\
&+ \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&\mu_{Rb,t} - r_{ft} = \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)\frac{\theta_t\omega_t - \omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t} \\
&+ \sigma_\xi\left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&\mu_{\omega t} = \left(\theta_t - 1\right)\left(\mu_{Rkt} - \mu_{Rb,t}\right) + \left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) \\
&+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right] \\
&\sigma_{\theta a,t} = -\left(\theta_t - 1\right)\left(\sigma_{ka,t} - \sigma_{ba,t}\right) \\
&\sigma_{\theta\xi,t} = -\left(\theta_t - 1\right)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right) \\
&\beta\left(\theta_t\omega_t - \omega_t\right)\sigma_{ba,t} = -\left(\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)\sigma_{ka,t} \\
&+ \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_a \\
&\beta\left(\theta_t\omega_t - \omega_t\right)\sigma_{b\xi,t} = -\left(\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)\sigma_{k\xi,t} \\
&\alpha^{-2}\theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 \\
&0 = \left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\left(1-\theta_t\omega_t\right)\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) \\
&+ \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)p_{kt}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\sigma_{ka,t}\right) \\
&- \beta p_{kt}\omega_t\mu_{\omega t} + \theta_t\omega_t p_{kt}^2\left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\
&- \beta p_{kt}\omega_t\left(\left(\sigma_{ka,t} - \sigma_a\right)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right).
\end{aligned}
$$

Notice that the first eight equations describe the evolutions of θ_t, ω_t, the return of risky intermediary debt R_{bt}, and the expected excess return to direct capital holding in terms of the two state variables, (θ_t, ω_t) and the loadings, $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$, of the return to direct capital holding on the two fundamental shocks in the economy.[10] The final two equations, then, express $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$ in terms of the state variables.

Before solving the final two equations, we simplify the equilibrium conditions. Notice first that

$$
\left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) = -\left(\theta_t - 1\right)\sigma_{ka,t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right) - \left(\theta_t - 1\right)\sigma_{k\xi,t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right),
$$

[10] Recall that we have also expressed the price of capital in terms of the state variables.

and

$$\mu_{Rkt} - \mu_{Rb,t} = \left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \sigma_{ka,t}\sigma_{ba,t} - \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t}$$
$$- \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2 - \sigma_{ka,t}\sigma_{ba,t} - \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}$$
$$+ \sigma_\xi\left(\left(\sigma_{ka,t} - \sigma_{ba,t}\right)\rho_{\xi,a} + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}\right).$$

Thus,

$$\left(\mu_{Rkt} - \mu_{Rb,t}\right) + \frac{1}{\theta_t - 1}\left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) = -\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right)^2 - \frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)^2$$
$$+ \sigma_\xi\left(\left(\sigma_{ka,t} - \sigma_{ba,t}\right)\rho_{\xi,a} + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}\right).$$

Using

$$\beta\left(\theta_t\omega_t - \omega_t\right)\left(\sigma_{ka,t} - \sigma_{ba,t}\right) = \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\sigma_{ka,t} - \sigma_a\right)$$
$$\beta\left(\theta_t\omega_t - \omega_t\right)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right) = \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_{k\xi,t}$$

we can thus express the drift of ω_t as

$$\mu_{\omega t} = -\frac{1}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2\left[\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right]$$
$$+ \frac{\sigma_\xi}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\left(\sigma_{ka,t} - \sigma_a\right)\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)$$
$$+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right].$$

Substituting the risk-based capital constraint, this becomes

$$\mu_{\omega t} = -\frac{1}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2\left[\sigma_a^2 - 2\sigma_a\sigma_{ka,t} + \frac{\theta_t^{-2}}{\alpha^2}\right]$$
$$+ \frac{\sigma_\xi}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\left(\sigma_{ka,t} - \sigma_a\right)\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)$$
$$+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right]$$
$$\equiv \mathcal{O}_0\left(\omega_t, \theta_t\right) + \mathcal{O}_a\left(\omega_t, \theta_t\right)\sigma_{ka,t} + \mathcal{O}_\xi\left(\omega_t, \theta_t\right)\sigma_{k\xi,t},$$

where

$$\mathcal{O}_0\left(\omega_t, \theta_t\right) = -\frac{1}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2\left[\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2}\right] \tag{B.2}$$
$$- \frac{\sigma_\xi\sigma_a\rho_{\xi,a}}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)$$

$$+\frac{1-\omega_t}{\omega_t}\left[\left(\rho_h-\frac{\sigma_\xi^2}{2}\right)-\frac{1}{p_{kt}}+\Phi\left(i_t\right)\theta_t\omega_t\right]$$

$$\mathcal{O}_a\left(\omega_t,\theta_t\right)=\frac{2\sigma_a}{\beta^2\omega_t}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2+\frac{\sigma_\xi\rho_{\xi,a}}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right) \tag{B.3}$$

$$\mathcal{O}_\xi\left(\omega_t,\theta_t\right)=\frac{\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}}{\beta\omega_t}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right). \tag{B.4}$$

Substituting into the drift rate of intermediary leverage

$$\begin{aligned}\mu_{\theta t}&=\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right)-\mu_{\omega t}+\sigma_{\theta a,t}^2+\sigma_{\theta\xi,t}^2\\&=\mathcal{S}_0\left(\omega_t,\theta_t\right)+\mathcal{S}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t}+\mathcal{S}_\xi\left(\omega_t,\kappa_t\right)\sigma_{k\xi,t},\end{aligned}$$

where

$$\mathcal{S}_0\left(\omega_t,\theta_t\right)=\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right)-\mathcal{O}_0\left(\omega_t,\theta_t\right)+\left(\frac{2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)}{\beta\omega_t}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \tag{B.5}$$

$$\mathcal{S}_a\left(\omega_t,\theta_t\right)=-2\sigma_a\left(\frac{2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)}{\beta\omega_t}\right)-\mathcal{O}_a\left(\omega_t,\theta_t\right) \tag{B.6}$$

$$\mathcal{S}_\xi\left(\omega_t,\theta_t\right)=-\mathcal{O}_\xi\left(\omega_t,\theta_t\right). \tag{B.7}$$

Similarly, the excess return on capital is given by

$$\begin{aligned}\mu_{Rk,t}-r_{ft}=&\left(\sigma_{ka,t}^2+\sigma_{k\xi,t}^2\right)\frac{1-\theta_t\omega_t}{1-\omega_t}+\left(\sigma_{ka,t}\sigma_{ba,t}+\sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\\&+\sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a}+\sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)\\=&\frac{\theta_t^{-2}}{\alpha^2}\frac{1-\theta_t\omega_t}{1-\omega_t}-\frac{\beta\left(1-\theta_t\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2}\\&+\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t}+\sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a}+\sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)\\=&\frac{-2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2}+\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t}\\&+\sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a}+\sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right).\end{aligned}$$

The excess return on intermediary debt is given by

$$\begin{aligned}\mu_{Rb,t}-r_{ft}=&\left(\sigma_{ba,t}^2+\sigma_{b\xi,t}^2\right)\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}+\left(\sigma_{ka,t}\sigma_{ba,t}+\sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t}\\&+\sigma_\xi\left(\sigma_{ba,t}\rho_{\xi,a}+\sigma_{b\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)\\=&\left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta\left(1-\theta_t\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\right)^2\frac{\theta_t^{-2}}{\alpha^2}\\&+\left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\right)^2\sigma_a^2\end{aligned}$$

$$
\begin{aligned}
&- 2\left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right)\left(\frac{\beta\left(1-\theta_t\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\right)\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
&- \left(\frac{1-\theta_t\omega_t}{\theta_t\omega_t-\omega_t}\right)\frac{\beta\left(1-\theta_t\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} \\
&+ \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\right)\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\sigma_a\sigma_{ka,t} \\
&+ \sigma_\xi\rho_{\xi,a}\left[-\frac{\beta\left(1-\theta_t\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\sigma_{ka,t}+\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\sigma_a\right] \\
&- \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\frac{\beta\left(1-\theta_t\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t-\omega_t\right)}\sigma_{k\xi,t}.
\end{aligned}
$$

Notice also that we can now derive the risk-free rate. Recall that, in the unconstrained region, the risk-free rate satisfies the household Euler equation

$$
r_{ft} = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \frac{1}{dt}\mathbb{E}\left[\frac{dc_t}{c_t}\right] - \frac{1}{dt}\mathbb{E}\left[\frac{\langle dc_t\rangle^2}{c_t^2} + \frac{\langle dc_t, d\xi_t\rangle^2}{c_t}\right].
$$

Applying Itô's lemma to the goods clearing condition, we obtain

$$
\begin{aligned}
dc_t &= d\left(K_tA_t - i_tk_tA_t\right) \\
&= A_t dK_t + \left(K_t - i_tk_t\right)dA_t - A_tk_tdi_t - A_ti_tdk_t - k_t\left\langle di_t, dA_t\right\rangle.
\end{aligned}
$$

From the financial intermediaries' optimal investment choice, we have

$$
\begin{aligned}
di_t &= \frac{\phi_0^2\phi_1p_{kt}^2}{2}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\left(\sigma_{ka,t}-\sigma_a\right)\right)dt \\
&+ \frac{\phi_0^2\phi_1p_{kt}^2}{4}\left(\left(\sigma_{ka,t}-\sigma_a\right)^2 + \sigma_{k\xi,t}^2\right)dt \\
&+ \frac{\phi_0^2\phi_1p_{kt}^2}{2}\left(\sigma_{ka,t}-\sigma_a\right)dZ_{at} + \frac{\phi_0^2\phi_1p_{kt}^2}{2}\sigma_{k\xi,t}dZ_{\xi t}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{dt}\mathbb{E}\left[\frac{dc_t}{c_t}\right] &= \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) \\
&- \left(\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\right)\frac{\phi_0^2\phi_1p_{kt}^2}{2}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right)\right) \\
&- \left(\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\right)\frac{\phi_0^2\phi_1p_{kt}^2}{4}\left(\left(\sigma_{ka,t}-\sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\
\frac{1}{dt}\mathbb{E}\left[\frac{\langle dc_t\rangle^2}{c_t^2}\right] &= \left(\sigma_a - \left(\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\right)\frac{\phi_0^2\phi_1p_{kt}^2}{2}\left(\sigma_{ka,t}-\sigma_a\right)\right)^2 \\
&+ \left(\left(\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\right)\frac{\phi_0^2\phi_1p_{kt}^2}{2}\sigma_{k\xi,t}\right)^2
\end{aligned}
$$

$$\frac{1}{dt}\mathbb{E}\left[\frac{\langle dc_t d\xi_t\rangle^2}{c_t}\right] = \sigma_\xi\left(\sigma_a - \left(\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\right)\frac{\phi_0^2\phi_1 p_{kt}^2}{2}\sigma_{ka,t}\right)\rho_{\xi,a}$$
$$-\sigma_\xi\left(\left(\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\right)\frac{\phi_0^2\phi_1 p_{kt}^2}{2}\sigma_{k\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}.$$

Recall that, in equilibrium, we have

$$1-i_t\theta_t\omega_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)p_{kt}\left(1-\omega_t\right),$$

so that

$$\frac{1}{1-i_t\theta_t\omega_t}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 = \left(\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)p_{kt}\left(1-\omega_t\right)\right)^{-1}\frac{\phi_0^2\phi_1}{4}p_{kt}^2$$
$$= \frac{p_{kt}}{\beta\left(1-\omega_t\right)}$$

and

$$1+\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 = \frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}.$$

Substituting into the expression for the risk-free rate, we obtain

$$\begin{aligned}
r_{ft} = &\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right)\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \\
&+ \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) - \frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}}\right) \\
&- \frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2} - 2\sigma_{ka,t}\sigma_a\right) - \left(\frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \\
&- \sigma_a^2\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)^2 + \frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_\xi\rho_{\xi,a}\sigma_{ka,t} \\
&+ \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
&- \sigma_a\sigma_\xi\rho_{\xi,a}\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right) + \frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t}.
\end{aligned}$$

We can now solve for the return on capital. In particular, we have

$$\begin{aligned}
\mu_{Rk,t} = &\, r_{ft} - \frac{2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} \\
&+ \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
= &\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right)\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right)-\frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t}-\frac{1}{p_{kt}}\right)\\
&-\frac{3\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2}+\frac{\beta\left(1-\omega_t\right)+4\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t}\\
&-\sigma_a^2\left(\frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}+\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)^2\right)\\
&+\frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a\\
&-\sigma_a\sigma_\xi\rho_{\xi,a}\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)-\left(\frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)^2\frac{\theta_t^{-2}}{\alpha^2}\\
&+\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_\xi\rho_{\xi,a}\sigma_{ka,t}\\
&+\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t}.
\end{aligned}
$$

Solving for $\mu_{Rk,t}$, we obtain

$$
\mu_{Rk,t}=\mathcal{K}_0\left(\omega_t,\theta_t\right)+\mathcal{K}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t}+\mathcal{K}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t},
$$

where

$$
\begin{aligned}
\mathcal{K}_0\left(\omega_t,\theta_t\right)=&\frac{\beta\left(1-\omega_t\right)}{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}\left(\rho_h-\frac{\sigma_\xi^2}{2}+\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right)\right) \quad \text{(B.8)}\\
&+\frac{2\theta_t\omega_t}{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}-\sigma_a\sigma_\xi\rho_{\xi,a}\\
&+\bar{a}+\frac{\sigma_a^2}{2}-\lambda_k-\frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}\left(3+\frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)\frac{\theta_t^{-2}}{\alpha^2}\\
&-\sigma_a^2\left(\frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}+\left(\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\right)\\
\mathcal{K}_a\left(\omega_t,\theta_t\right)=&\sigma_\xi\rho_{\xi,a}+\frac{\beta\left(1-\omega_t\right)+4\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}\sigma_a+\frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_a \quad \text{(B.9)}\\
\mathcal{K}_\xi\left(\omega_t,\theta_t\right)=&\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}. \quad \text{(B.10)}
\end{aligned}
$$

We can now express the risk-free rate in the economy as

$$
\begin{aligned}
r_{ft}=&\mu_{Rk,t}+\frac{2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2}-\frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t}\\
&-\sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a}+\sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right)\\
\equiv&\mathcal{R}_0\left(\omega_t,\theta_t\right)+\mathcal{R}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t},
\end{aligned}
$$

where

$$
\mathcal{R}_0\left(\omega_t,\theta_t\right)=\mathcal{K}_0\left(\omega_t,\theta_t\right)+\frac{2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} \quad \text{(B.11)}
$$

$$\mathcal{R}_a(\omega_t,\theta_t) = \mathcal{K}_a(\omega_t,\theta_t) - \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\sigma_a - \sigma_\xi\rho_{\xi,a}. \tag{B.12}$$

Substituting into the excess return on holding intermediary debt, we obtain

$$\mu_{Rb,t} = \mathcal{B}_0(\omega_t,\theta_t) + \mathcal{B}_a(\omega_t,\theta_t)\sigma_{ka,t} + \mathcal{B}_\xi(\omega_t,\theta_t)\sigma_{k\xi,t},$$

where

$$\begin{aligned}
\mathcal{B}_0(\omega_t,\theta_t) &= \mathcal{R}_0(\omega_t,\theta_t) + \left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \qquad \text{(B.13)}\\
&\quad + \left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)^2\sigma_a^2\\
&\quad + \sigma_\xi\rho_{\xi,a}\sigma_a\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\\
&\quad - \left(\frac{1-\theta_t\omega_t}{\theta_t\omega_t-\omega_t}\right)\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2}\\
\mathcal{B}_a(\omega_t,\theta_t) &= \mathcal{R}_a(\omega_t,\theta_t) + \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\right)\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\sigma_a \qquad \text{(B.14)}\\
&\quad - 2\left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\sigma_a\\
&\quad - \sigma_\xi\rho_{\xi,a}\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\\
\mathcal{B}_\xi(\omega_t,\theta_t) &= \mathcal{R}_\xi(\omega_t,\theta_t) - \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}. \qquad \text{(B.15)}
\end{aligned}$$

Notice that

$$p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\mathcal{K}_\xi(\omega_t,\theta_t) - \beta p_{kt}\omega_t\mathcal{O}_\xi(\omega_t,\theta_t) = 0.$$

Using these results and the risk-based capital constraint, we can rewrite

$$\begin{aligned}
0 &= \theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right)\\
&\quad + p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a(\sigma_{ka,t}-\sigma_a)\right)\\
&\quad - \beta p_{kt}\omega_t\mu_{\omega_t} - p_{kt}(\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left((\sigma_{ka,t}-\sigma_a)^2 + \sigma_{k\xi,t}^2\right)
\end{aligned}$$

as

$$\begin{aligned}
0 &= \theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right)\\
&\quad + p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a(\sigma_{ka,t}-\sigma_a)\right)\\
&\quad - \beta p_{kt}\omega_t\mu_{\omega_t} - p_{kt}(\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 - 2\sigma_{ka,t}\sigma_a\right)
\end{aligned}$$

$$\equiv \mathcal{C}_0(\omega_t,\theta_t) + \mathcal{C}_a(\omega_t,\theta_t)\,\sigma_{ka,t},$$

where

$$\begin{aligned}
\mathcal{C}_0(\omega_t,\theta_t) &= \theta_t\omega_t(1-\theta_t\omega_t)\,\Phi(i_t)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \qquad \text{(B.16)}\\
&\quad - p_{kt}(\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right)\\
&\quad + p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left(\mathcal{K}_0(\omega_t,\theta_t) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k\right)\\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_0(\omega_t,\theta_t)\\
\mathcal{C}_a(\omega_t,\theta_t) &= p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\,\mathcal{K}_a(\omega_t,\theta_t) \qquad \text{(B.17)}\\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_a(\omega_t,\theta_t) + p_{kt}\beta(1-\omega_t)\,\sigma_a.
\end{aligned}$$

Solving for $\sigma_{ka,t}$, we obtain

$$\sigma_{ka,t} = -\frac{\mathcal{C}_0(\omega_t,\theta_t)}{\mathcal{C}_a(\omega_t,\theta_t)}.$$

Substituting into the risk-based capital constraint, we obtain

$$\frac{\theta_t^{-2}}{\alpha^2} = \sigma_{k\xi,t}^2 + \left(\frac{\mathcal{C}_0(\omega_t,\theta_t)}{\mathcal{C}_a(\omega_t,\theta_t)}\right)^2,$$

so that

$$\sigma_{k\xi,t} = \sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \left(\frac{\mathcal{C}_0(\omega_t,\theta_t)}{\mathcal{C}_a(\omega_t,\theta_t)}\right)^2}.$$

We can further simplify the above expressions by substituting for $\mathcal{O}_0$, $\mathcal{O}_a$, $\mathcal{K}_0$, and $\mathcal{K}_a$. Notice first that

$$\begin{aligned}
\mathcal{C}_a(\omega_t,\theta_t) &= p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\,\mathcal{K}_a(\omega_t,\theta_t)\\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_a(\omega_t,\theta_t) + p_{kt}\beta(1-\omega_t)\,\sigma_a\\
&= p_{kt}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\left(\mathcal{K}_a(\omega_t,\theta_t) - \sigma_\xi\rho_{\xi,a}\right)\\
&\quad + p_{kt}\left\{-\frac{2\sigma_a}{\beta}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))^2 + \beta(1-\omega_t)\,\sigma_a\right\}\\
&= p_{kt}\left(\beta(1-\omega_t) + 4\theta_t\omega_t p_{kt} + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\right)\sigma_a\\
&\quad + p_{kt}\left\{-\frac{2\sigma_a}{\beta}(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))^2 + \beta(1-\omega_t)\,\sigma_a\right\}\\
&= \frac{2\sigma_a p_{kt}}{\beta}\left(\frac{\omega_t}{1-\omega_t}\right)(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{C}_0\left(\omega_t, \theta_t\right) &= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&- p_{kt}\left(\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right) \\
&+ p_{kt}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\mathcal{K}_0\left(\omega_t, \theta_t\right) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k\right) \\
&- \beta p_{kt}\omega_t\mathcal{O}_0\left(\omega_t, \theta_t\right) \\
&= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&- p_{kt}\left(\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right) \\
&+ \beta p_{kt}\left(1-\omega_t\right)\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) \\
&- \theta_t\omega_t p_{kt}^2\left(3 + \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)\frac{\theta_t^{-2}}{\alpha^2} \\
&- \sigma_a^2 p_{kt}\left(\theta_t\omega_t p_{kt} + \frac{\left(\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)^2}{\beta\left(1-\omega_t\right)}\right) \\
&+ \frac{p_{kt}}{\beta}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2\left(\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2}\right) - \beta p_{kt}\left(1-\omega_t\right)\Phi\left(i_t\right)\theta_t\omega_t
\end{aligned}$$

Collecting like terms, we obtain

$$\begin{aligned}
\mathcal{C}_0\left(\omega_t, \theta_t\right) &= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&- \beta p_{kt}\left(1-\omega_t\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1 - \frac{1-i_t}{1-i_t\theta_t\omega_t}\right) \\
&+ \frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\left\{\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2 - \beta\left(\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\right\} \\
&- \frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\theta_t\omega_t p_{kt}\left(3\beta + \frac{4p_{kt}\theta_t\omega_t}{1-\omega_t}\right) \\
&+ \sigma_a^2\frac{p_{kt}}{\beta}\left\{\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2 - \beta\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\right\} \\
&- \sigma_a^2\frac{p_{kt}}{\beta}\frac{\left(\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)^2}{\left(1-\omega_t\right)} \\
&= -\frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\frac{\omega_t}{1-\omega_t}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2 \\
&- \sigma_a^2\frac{p_{kt}}{\beta}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\frac{\omega_t\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right) + 1 - \omega_t}{1-\omega_t}
\end{aligned}$$

Thus

$$\begin{aligned}\sigma_{ka,t} &= -\frac{\mathcal{C}_0\left(\omega_t,\theta_t\right)}{\mathcal{C}_a\left(\omega_t,\theta_t\right)}\\ &= \frac{\theta_t^{-2}}{\alpha^2}+\sigma_a^2\left(1+\frac{1-\omega_t}{\omega_t\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)}\right).\end{aligned}$$

C Constant leverage benchmark

We begin by solving for the equilibrium dynamics of the intermediaries' wealth share in the economy. Recall that the capital held by the intermediaries is given by

$$k_t = \theta\omega_t K_t.$$

Applying Itô's lemma, we obtain

$$\begin{aligned}\frac{dk_t}{k_t} &= \frac{d\omega_t}{\omega_t}+\frac{dK_t}{K_t}\\ &= \left(\mu_{\omega t}+\Phi\left(i_t\right)\theta\omega_t-\lambda_k\right)dt+\sigma_{\omega a,t}dZ_{at}+\sigma_{\omega\xi,t}dZ_{\xi,t}.\end{aligned}$$

Recall, on the other hand, that intermediary capital evolves as

$$\frac{dk_t}{k_t} = \left(\Phi\left(i_t\right)-\lambda_k\right)dt.$$

Thus, equating coefficients, we obtain

$$\begin{aligned}\sigma_{\omega a,t} &= 0\\ \sigma_{\omega\xi,t} &= 0\\ \mu_{\omega t} &= \Phi\left(i_t\right)\left(1-\theta\omega_t\right).\end{aligned}$$

Consider now the wealth evolution of the representative household. From the households' budget constraint, we have

$$\frac{dw_{ht}}{w_{ht}} = \left(r_{ft}-\rho_h+\frac{\sigma_\xi^2}{2}\right)dt+\frac{1-\bar{\theta}\omega_t}{1-\omega_t}\left(dR_{kt}-r_{ft}dt\right)+\frac{\omega_t\left(\bar{\theta}-1\right)}{1-\omega_t}\left(dR_{bt}-r_{ft}dt\right).$$

On the other hand, from the definition of ω_t, we obtain

$$\begin{aligned}\frac{dw_{ht}}{w_{ht}} &= \frac{d\left(\left(1-\omega_t\right)p_{kt}A_tK_t\right)}{\left(1-\omega_t\right)p_{kt}A_tK_t}\\ &= \frac{dp_{kt}}{p_{kt}}+\frac{dA_t}{A_t}+\frac{dK_t}{K_t}-\frac{\omega_t}{1-\omega_t}\frac{d\omega_t}{\omega_t}+\left\langle\frac{dp_{kt}}{p_{kt}},\frac{dA_t}{A_t}\right\rangle.\end{aligned}$$

Equating coefficients once again and simplifying, we obtain

$$\begin{aligned}\sigma_{ba,t} &= \sigma_{ka,t}\\ \sigma_{b\xi,t} &= \sigma_{k\xi,t}\end{aligned}$$

$$\mu_{Rb,t} = \mu_{Rk,t} + \frac{1-\omega_t}{\omega_t\left(\bar{\theta}-1\right)}\left(\rho_h - \frac{\sigma_\xi^2}{2} - \frac{1}{p_{kt}}\right) + \Phi\left(i_t\right).$$

We now turn to solving for the equilibrium price of capital. The goods clearing condition in this economy reduces to

$$\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(1-\omega_t\right)p_{kt} = 1 - i_t\bar{\theta}\omega_t.$$

Substituting the optimal level of investment

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right),$$

we obtain that the price of capital satisfies

$$\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(1-\omega_t\right)p_{kt} = 1 - \frac{\bar{\theta}\omega_t}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

Then the price of capital satisfies

$$0 = \bar{\theta}\omega_t p_{kt}^2 + \beta\left(1-\omega_t\right)p_{kt} - \frac{4}{\phi_0^2\phi_1^2}\left(\bar{\theta}\omega_t + \phi_1\right),$$

or:

$$p_{kt} = \frac{-\beta\left(1-\omega_t\right) + \sqrt{\beta^2\left(1-\omega_t\right)^2 + \frac{16\bar{\theta}\omega_t}{\phi_0^2\phi_1^2}\left(\bar{\theta}\omega_t + \phi_1\right)}}{2\bar{\theta}\omega_t}.$$

Applying Itô's lemma, we obtain

$$0 = \bar{\theta}\omega_t p_{kt}^2\left(2\frac{dp_{kt}}{p_{kt}} + \left\langle\frac{dp_{kt}}{p_{kt}}\right\rangle^2 + \frac{d\omega_t}{\omega_t}\right) + \beta\left(1-\omega_t\right)p_{kt}\frac{dp_{kt}}{p_{kt}} - \frac{\omega_t}{1-\omega_t}\beta p_{kt}\frac{d\omega_t}{\omega_t} + \frac{4}{\phi_0^2\phi_1^2}\bar{\theta}\frac{d\omega_t}{\omega_t}.$$

Equating coefficients and simplifying, we obtain

$$\begin{aligned}
\sigma_{ka,t} &= \sigma_a \\
\sigma_{k\xi,t} &= 0 \\
\mu_{Rk,t} &= \frac{1}{p_{kt}} + \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k \\
&\quad - \frac{\Phi\left(i_t\right)\left(1-\bar{\theta}\omega_t\right)}{p_{kt}\left(2\bar{\theta}\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)}\left(\frac{4\bar{\theta}}{\phi_0^2\phi_1^2} - \frac{\omega_t}{1-\omega_t}\beta p_{kt} + \bar{\theta}\omega_t p_{kt}^2\right).
\end{aligned}$$

Finally, consider the equilibrium risk-free rate. Notice that

$$\frac{dc_t}{c_t} = \frac{d\left(\left(1-i_t\bar{\theta}\omega_t\right)A_tK_t\right)}{\left(1-i_t\bar{\theta}\omega_t\right)A_tK_t}$$

$$= \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\bar{\theta}\omega_t}{1 - i_t\bar{\theta}\omega_t} di_t - \frac{i_t\bar{\theta}\omega_t}{1 - i_t\bar{\theta}\omega_t}\frac{d\omega_t}{\omega_t} - \frac{\bar{\theta}\omega_t}{1 - i_t\bar{\theta}\omega_t}\left\langle di_t, \frac{dA_t}{A_t}\right\rangle,$$

and

$$di_t = d\left(\frac{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}{\beta} p_{kt}^2 - \frac{1}{\phi_1}\right) = \frac{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}{\beta} p_{kt}^2 \left(2\frac{dp_{kt}}{p_{kt}} + \left\langle \frac{dp_{kt}}{p_{kt}}\right\rangle^2\right).$$

Using

$$1 - i_t\bar{\theta}\omega_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)(1 - \omega_t)\, p_{kt},$$

the risk-free rate is thus given by

$$\begin{aligned} r_{ft} &= \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \frac{1}{dt}\mathbb{E}_t\left[\frac{dc_t}{c_t}\right] - \frac{1}{dt}\mathbb{E}_t\left[\left\langle\frac{dc_t}{c_t}\right\rangle\right] \\ &= \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \bar{a} - \frac{\sigma_a^2}{2} + \Phi\left(i_t\right)\bar{\theta}\omega_t - \lambda_k \\ &\quad - \frac{2\bar{\theta}\omega_t p_{kt}}{\beta\left(1 - \omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2}\right) - \frac{i_t\bar{\theta}\omega_t}{1 - i_t\bar{\theta}\omega_t}\mu_{\omega t}. \end{aligned}$$