Optimal policy with heterogeneous agents and aggregate shocks: An application to optimal public debt dynamics

François Le Grand Xavier Ragot*

March 25, 2016

Abstract

We show that allocations in incomplete insurance-market economies can be represented as the solution of the program of a constrained planner. This representation allows for solving Ramsey programs in incomplete-market economies with aggregate shocks, and thus determining optimal policies in such setups. We apply this framework to derive optimal public debt and fiscal policy after a technology, a government spending or an uncertainty shock. We find that, for any adverse shock, public debt decreases whereas capital taxes increases on impact. This policy limits the fall in capital after such shocks. Simulations of the optimal solutions can be obtained by simple perturbation methods.

Keywords: Incomplete markets, optimal policy, public debt.

JEL codes: E21, E44, D91, D31.

*François Le Grand: EMLyon Business School `legrand@em-lyon.com`; Xavier Ragot: CNRS, OFCE and PSE `xavier.ragot@gmail.com`. Corresponding author: Xavier Ragot.

1 Introduction

Incomplete insurance-market models with aggregate shocks are environments which allow thinking about the dynamics of heterogeneity and inequality in general equilibrium. Such models can improve our understanding of optimal policy designs, by considering rich but relevant tradeoffs between redistribution, insurance and incentives. In these models, agents face incomplete markets and borrowing limits that prevent them from perfectly insuring themselves against idiosyncratic risk, in the tradition of the Bewley-Huggett-Aiyagari literature (Bewley, 1983; Huggett 1993 and Aiyagari 1994). Unfortunately, these models are known to be hard to solve, from both an analytical or a numerical point of view, because of the time-varying heterogeneity across agents (Krusell and Smith, 1998). Due to these difficulties, no analysis of the distortions and optimal policies in these environments with aggregate shocks have been conducted yet. In particular, optimal Ramsey programs have not been solved. In this paper, we propose a tractable representation of incomplete insurance market economies with aggregate shocks, which allows deriving and simulating such optimal policies. We apply this framework to derive optimal fiscal policy and optimal debt management, and in particular their response to various types of shocks. We can therefore discuss tradeoffs between insurance and efficiency in such environments.

The construction of our tractable framework relies on the following observation. When insurance markets are incomplete for idiosyncratic risks, agents with different individual risk realizations have different wealth and consumption levels. As a consequence, even if agents are initially identical, there is an increasing number of heterogeneous wealth levels, as time goes by. Huggett (1993) and Aiyagari (1994) have shown that these economies have a recursive structure in absence of aggregate shocks, provided that a continuous distribution of wealth is introduced as a state variable. We construct an environment where the heterogeneity across agents depends only on a finite, but possibly large, number of consecutive past realizations of the idiosyncratic risk (as a theoretical outcome). In our model, for a given (but potentially arbitrarily large) N, agents with the same idiosyncratic risk realizations for the previous N periods choose sthe same consumption and wealth. As a consequence, instead of having a continuous distribution of heterogeneous agents in each period, the economy is characterized by a finite number of heterogeneous consumption and wealth levels.

The interest of this framework relies on the five following properties. First, the equilibrium allocation is determined as the solution of a central planner allocation, where the planner faces a set of explicit constraints on its ability to transfer resources across agents. We can thus derive a recursive formulation of our equilibrium with aggregate shocks, which simplifies the derivation of equilibrium conditions.

Second, we show that the central planner allocation characterizing our equilibrium can be expressed as the solution of a decentralized optimization program. More precisely, we exhibit a transfer scheme in a decentralized market economy with aggregate shocks, such that the competitive equilibrium exactly coincides with the equilibrium of the planner economy.

Third, we show that the allocations of the planner economy and of the decentralized economy coincide with each other, not only for any finite $N-$period individual histories, but also for infinitely long histories, i.e., when $N \to \infty$, provided that certain conditions hold. As a consequence, the equilibrium of the planner economy can be made arbitrarily close to general competitive equilibrium (with incomplete insurance markets as studied in Krusell and Smith, 1998) whenever these conditions are fulfilled. As an additional result, this proves the existence of a policy rule which depends on the distribution of wealth and current idiosyncratic state and wealth level, as in Krusell and Smith, \citeyear*{KrSm:98}.

Fourth, using the finite-state equilibrium representation, we can solve the Ramsey program with aggregate shocks and derive optimal policies in a time-varying environment. This enables us to obtain responses of policies to various shocks, such as a technology, a government spending, an uncertainty shock.

Finally, the finite equilibrium structure simplifies to a large extent the simulation of our model. Instead of having to track a continuous distribution of agents for wealth and consumptions, we only need to account for a finite (even if possibly large) number of different agent types, where all agents of the same type have the same consumption level and the same asset holdings. We can therefore use standard numerical techniques for the equilibrium simulation. In particular, provided that shocks are not too large, standard perturbation methods can be used.

We apply this framework to investigate the design of optimal fiscal policy in an incomplete market environment, when the government can issue public debt and must rely on distorting taxes on capital and labor to finance public spending. This framework can first be seen as a extension to an incomplete insurance market environment of standard optimal taxation models with a representative agent (Stockey and Lucas, 1983; Aiyagari, Marcet, Sargent and Seppala

2002; Farhi 2012 among others). Second, it can be seen as the solution of a Ramsey program in a rather standard incomplete market economy with aggregate shocks, first studied by Krusell and Smith (1998) and analyzed in Heathcote (2005) (see the literature review below for references to recent and important contributions). Considering incomplete insurance markets deeply affects optimal public debt determination. Indeed, agents use both public debt and the capital stock for self-insurance motives, which generates a well-defined optimal level of public debt, when there is no aggregate shocks (Aiyagari and McGrattan, 1998 and Acikgoz, 2013 for a recent analysis).

In this environment, we consider three types of aggregate shocks. The first one is a technology shock, which affects the productivity of both capital and labor. The second one is a shock to the persistence of the idiosyncratic risk, which can be interpreted as an uncertainty or inequality shock. The third one is a shock on public spending that affects the financing needs of the government. We find that public debt and taxes are mean-reverting after any transitory aggregate shock. In addition, when the economy is hit by a negative shock, we find that optimal taxes increase to reduce public debt. This optimal outcome tends to reduce the fall in aggregate capital after a negative shock. The amplitude of this general pattern varies with the shock considered. In addition, we find that capital taxes are more volatile than labor taxes, what is also a result in models featuring a representative agent (Aiyagari, Marcet, Sargent and Seppala, 2002, among others). Finally, we find that capital taxes generally increase after a negative shock.

This paper contributes first to the literature on the theory of incomplete insurance-market economies with aggregate shocks. We propose a tractable model which generalizes a number of previous works. Some environments have been provided to show equilibrium existence and to derive theoretical properties when insurance markets are incomplete. First, a class of no-trade equilibria has been studied, relying of permanent idiosyncratic shocks (Constantinides and Duffie, 1996). This framework is used for instance in Heathcote, Storesletten and Violante (2014). Krusell, Mukoyama and Smith (2011) study a class of no-trade equilibrium in an economy without capital and with a tight-enough credit constraint. Recently, a class of small-trade equilibrium has been provided, based on quasi-linear utility function (Challe, Le Grand and Ragot, 2013; Challe and Ragot, 2014 or Le Grand and Ragot, 2015).[1] While these models

[1]Quasi-linearity has been used in other setups to reduce heterogeneity. Lagos and Wright (2005) consider a model with linear disutility in labor in order to reduce heterogeneity in the time-dimension (every agent has the same marginal utility of consumption at the end of each period). Kiyotaki and Moore (2005, 2008), and Miao and Wang (2015) consider that entrepreneurs face idiosyncratic investment opportunities with a constant

allow for the derivation of the properties of the equilibrium allocation, they rely on specific assumptions about the shape of the utility function or the volume of assets. The environment we propose here enables us to consider general utility functions and arbitrary asset quantities. In addition, we can prove that a recursive equilibrium exists when the wealth of agents is introduced as a state variable, as in Krusell and Smith (1998) simulation strategy (see Miao, 2006, for a discussion).

Second, our paper contributes to the literature on optimal policies in incomplete insurance-market models without aggregate shocks (Aiyagari, 1995; Aiyagari and McGrattan, 1998). Recent papers, such as Acikgoz (2013), solve for the Ramsey program in these economies without aggregate shocks. Extending the analysis to economies with aggregate shocks allows considering a wide set of new economic problems, such as temporary change in uninsurable risk, or in technology.

Finally, this paper contributes to the vast literature on optimal time-varying fiscal policy, when the planner has distorting instruments, and cannot thus simply implement the first-best allocation. Seminal contributions consider a complete-market representative agents (Barro, 1979; Lucas and Stockey, 1983). More recent contributions consider incomplete market for the aggregate risk, introducing non state-contingent public debt (Aiyagari, Marcet, Sargent and Seppala, 2002; Farhi, 2012). Recently, Bhandari, Evans, Golosov and Sargent (2013) consider an environment with a finite number of agents facing idiosyncratic risk. They consider a Ramsey problem where the planner has access to non-distorting tools (positive or negative lump-sum transfers) and show that in this environment, public debt is irrelevant, which is a form of Ricardian equivalence. In the current paper, we do not allow the planner to use lump-sum transfers, what makes the level of public a relevant policy variable.

The rest of the paper is organized as follows. In Section 2, we present our economy and in particular the structure of aggregate and idiosyncratic shocks. We describe the central planner problem and derives the associated allocation in Section 3. We then show in Section 4 how the central planner allocation can be decentralized by a well-chosen lump-sum tax-system. We then take advantage in Section 5 of the finite equilibrium structure to solve the Ramsey program. Finally, conclusions are given in Section 7.

return-to-scale. This constant marginal productivity of idiosyncratic investments reduces the heterogeneity among entrepreneurs. Dang, Holmstrom, Gorton, and Ordoñez (2014) introduce a piecewise linear utility function to model the urgency to consume a certain amount of goods.

2 The environment

We consider a discrete-time economy populated by a continuum of agents, distributed on a segment J following a non-atomic measure ℓ. The segment is of length 1: $\ell(J) = 1$.[2] In each period t, the economy has two goods: a consumption-capital good and labor. The period utility function, denoted $U(c,l)$ is increasing and concave in consumption c, decreasing and convex in labor l and is twice continuously differentiable. The derivative with respect to consumption and labor are respectively denoted U_c and U_l. The marginal disutility of labor is null for zero labor: $U_l(c,0) = 0$. As is standard in the incomplete-insurance market literature, we consider a special class of utility function which does not exhibit wealth effect for the labor supply. We consider GHH utility function of the form:[3]

$$U(c,l) = u\left(c - \frac{l^{1+1/\varphi}}{1+1/\varphi}\right), \tag{1}$$

where $\varphi > 0$ and the function u is assumed to be continuously twice derivable, increasing and concave. The discount factor is $\beta \in (0,1)$. Each household is an expected utility maximizer, which ranks consumption and labor streams, denoted respectively as $(c_t)_{t\geq 0}$ and $(l_t)_{t\geq 0}$, according to $\sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$.

2.1 Aggregate risk and production

Time is discrete, indexed by $t \geq 0$. The aggregate risk is represented by a probability space $(\mathcal{S}^\infty, \mathcal{F}, \mathbb{P})$. The state space $\mathcal{S} \subset \mathbb{R}^+$ captures the aggregate shock. In any period t, the aggregate shock denoted s_t takes values in the state space $\mathcal{S}$. We assume that the aggregate state at date 0, denoted $s_0 \in \mathcal{S}$, is given. The history of aggregate shocks up to date t is summarized by the sequence of aggregates shocks from date 1 to dare t and denoted $s^t \equiv \{s_1, \ldots, s_t\} \in \mathcal{S}^t$.

[2] Among others, Feldman and Gilles (1985) have identified issues when applying the law of large number to a continuum of random variables. Green (1994) describes a construction of the sets J_i and of the measures ℓ_i to ensure that our statements hold. Feldman and Gilles (1985), Judd (1985), and Uhlig (1996) also propose other solutions to this issue. From now on, we assume that the law of large numbers applies.

[3] Although we focus on this utility function which does not exhibit wealth effect on the labor (see below), results are valid for general utility functions.

2.2 Idiosyncratic risk

Agents face time-varying idiosyncratic risk. At the beginning of each period, agents face an idiosyncratic labor productivity shock $e_t \in \mathcal{E} \equiv \{0, \ldots, E\} \in \mathbb{R}_+^{L+1}$ that follows a discrete first-order Markov process with transition matrix $M(s_t)$. The probability $M_{e,e'}(s_t)$, $e, e' = 0, \ldots, E$ is the probability for an agent to switch from individual state $e_t = e$ at date t to state $e_{t+1} = e'$ at date $t+1$, when aggregate state is s_t in period t. To cover the various cases that can be found in the literature, we assume that households in state $e \in \{1, \ldots, E\} = \mathcal{E} - \{0\}$ have a labor productivity n_e, whereas households in state $e = 0$ have a zero labor productivity but earn a home production δ. The former agents can be considered as employed workers with various levels of productivity, while the latter can be considered as unemployed workers.

At period t, we denote as $\eta_{e,t}(s^t) \in [0, 1]$ the share of the population in state $e \in \mathcal{E}$ and $s_t \in \mathcal{S}^t$. For instance, $\eta_{0,t}(s^t)$ is the share of the population in idiosyncratic state 0 at period t. Assuming that the law of large numbers holds in a continuum (see Footnote 2), these shares evolve as, for any $t \geq 1$:

$$\eta_{e,t}(s^t) = \sum_{e'=0}^{E} M_{e,e'}(s_{t-1})\eta_{e',t-1}(s^{t-1}),\ s^t \succeq s^{t-1}. \tag{2}$$

Remark 1 (Notations) *For the sake of clarity, we will lighten notations from now on. For any random variable $X_t : \mathcal{S}^t \to \mathbb{R}$, we will simply denote X_t instead of $X_t(s^t)$ a realization of the random variable X_t in state s^t. By the same token, for any random variable $Y_t : \mathcal{E}^t \times \mathcal{S}^t \to \mathbb{R}$, we will denote Y_{t,e^t} realization of the random variable Y_t in state (e^t, s^t).*

2.3 Production and assets

In any period $t \geq 1$, a production technology with constant-returns-to-scale (CRS) $F(K_{t-1}, L_t, s_{t-1})$ transforms capital K_{t-1} and labor L_t into output. Capital must be installed one period before production, and the productivity may depend on the aggregate state at the date of capital installation. Labor L_t is measured in efficient units. This formulation allows for capital depreciation, which is subsumed by the production function $F(\cdot, \cdot, \cdot)$. The production function $(K, L) \mapsto F(K, L, s)$ is smooth in K and L and satisfies the standard Inada conditions. The good is produced by a profit-maximizing representative firm. We denote as $\tilde{w}_t$ the before-tax real wage rate in period t and as $\tilde{r}_t$ the real before-tax rental rate of capital between period $t-1$ and period t. Profit maximization yields in each period $t \geq 1$:

$$\tilde{r}_t = F_K(K_{t-1}, L_t, s_{t-1}), \tag{3}$$

$$\tilde{w}_t = F_L(K_{t-1}, L_t, s_{t-1}). \tag{4}$$

Importantly, the previous assumptions imply that the return on the capital stock in period t is known in period $t-1$. Indeed, (i) the productivity of firms is known in period $t-1$, (ii) the labor transition process between period $t-1$ and period t is known in period $t-1$ and (iii) the labor supply exhibits no wealth effect.[4] Period t total labor supply is thus known in period $t-1$, as capital and labor productivity.

In each period t, the government has to finance a public good G_t, which is possibly stochastic. The public good can be financed either by levying distorting taxes on capital income τ_t^k or on labor income τ_t^l or by issuing an amount B_t of a riskless one period public debt.[5] Since the rate of return on capital at date t is known at date $t-1$, the absence of arbitrage implies that public debt and capital have to pay the same pre-tax interest rate $\tilde{r}_t$ for any aggregate history $s^t \in \mathcal{S}^t$. Capital income are made of interest payments of all interest-bearing assets, made of both capital and public debt. Interest payments amount to $\tilde{r}_t(K_{t-1} + B_{t-1})$, that are taxed with the same rate τ_t^k. Labor income are equal to the real wage multiplied by aggregate labor supply, $\tilde{w}_t L_t$, that is taxed at the rate τ_t^l. Debt issuance mechanically implies that in addition to the public good, the government also needs to reimburse the maturing public debt and the attached interest, which equals $(1 + \tilde{r}_t)B_{t-1}$.

We deduce that the period t budget constraint of the government can be expressed as follows:

$$G_t + (1 + \tilde{r}_t)B_{t-1} \leq \tau_t^l \tilde{w}_t L_t + \tau_t^k \tilde{r}_t (K_{t-1} + B_{t-1}) + B_t. \tag{5}$$

We now denote as $r_t = (1 - \tau_t^k)\tilde{r}_t$ and $w_t = (1 - \tau_t^l)\tilde{w}_t$ the after-tax real interest and real wage rate respectively. Using the CRS property of the production function that implies that $F(K, L, s) = KF_K(K, L, s) + NF_L(K, L, s)$, we deduce that the budget constraint (5) can be

[4] These points will be made explicit later on, in Lemma 1.

[5] The question of the optimal mix of these financing tools will be the focus of the second part of the paper and in particular of the Ramsey program studied in Section 5.

simplified as follows

$$G_t + r_t K_{t-1} + w_t L_t + (1 + r_t) B_{t-1} \leq F(K_{t-1}, L_t, s_{t-1}) + B_t. \tag{6}$$

Finally, if C_t^{tot} denotes the total agents' consumption in period t, the resource constraint of the economy can be expressed as follows

$$G_t + C_t^{tot} + K_t \leq F(K_{t-1}, L_t, s_{t-1}) + \eta_{0,t}\delta, \tag{7}$$

In equation (7) the right-hand side collects all resources, which are made of produced goods $F(K_{t-1}, L_t, s_{t-1})$ and of goods obtained from home production $\eta_{0,t}\delta$, where $\eta_{0,t}$ is the size of the population producing δ. The left hand side gathers uses, made of public spending, total consumption and capital.

3 The quasi-planner economy

In general, the previous economy features a growing heterogeneity over time, because agents with different realizations for the uninsurable idiosyncratic risk will choose different consumption levels and different amounts of wealth. The heterogeneity can be represented by a time-varying continuous distribution of wealth, which prevents any analytical characterization of the equilibrium and raises considerable computational challenges. We first present our environment in which the agents heterogeneity is summarized by only a finite number of agents types. A planner will allocate the resources among agents while facing some constraints to transfer resources across agents. Due to these constrains, we qualify the planner of *quasi-planner*. In Section 4, we show why this environment is interesting, and notably how it can be decentralized.

3.1 The islands

Our environment relies on the concept of islands. In a nutshell, island enables resource pooling and risk sharing for agents living on the same island, but prevents transfers across agents belonging to different islands. An island will be defined as the gathering of all agents sharing the same history for the last N periods. We denote as $e^N \in \mathcal{E}^N$ an history of individual risk realizations of length N. The set of possible individual risk realizations $\mathcal{E}$ being of cardinal

$E+1$, there are therefore $(E+1)^N$ different islands, indexed by $e^N \in \mathcal{E}^N$. At the beginning of every period, an agent on island $\hat{e}^N \in \mathcal{E}^N$ faces an idiosyncratic shock $e \in \mathcal{E}$ and will therefore be endowed with the history $e^N \in \mathcal{E}^N$. The agent will transit from island $\hat{e}^N$ to the island e^N and will take his own resources of island $\hat{e}^N$ when transferring to island e^N.

The quasi-planner aims at maximizing an utilitarian welfare criterion, i.e. the sum of individual welfares. The key assumption is that the central planner can transfer resources between agents of the same island, but cannot transfer resources across islands. The utilitarian welfare criterion implies that the quasi-planner treats agents symmetrically on each island and will allocate the same consumption level and the same end-of-period saving to all agents of the same island. The quasi-planner thus chooses the consumption level $\left(c_t : \mathcal{E}^N \times \mathcal{S}^t \to \mathbb{R}\right)_{t\geq 0}$, the labor supply $\left(l_t : \mathcal{E}^N \times \mathcal{S}^t \to \mathbb{R}\right)_{t\geq 0}$ and the end-of-period saving $\left(a_t : \mathcal{E}^N \times \mathcal{S}^t \to \mathbb{R}\right)_{t\geq 0}$ of agents in every island $e^N \in \mathcal{E}^N$ and in every state $s^t \in \mathcal{S}^t$, at all dates t. We assume that the quasi-planner faces a borrowing constraint $a_t(e^N, s^t) \geq -\bar{a}$ in every island $e^N \in \mathcal{E}^N$ and in every state $s^t \in \mathcal{S}^t$. This important assumption will further limit the ability of the quasi-planner to share risks across agents. Consistently with Remark 1, we will denote x_{t,e^N} for $e^N \in \mathcal{E}^N$ a realization of $x_t : \mathcal{E}^N \times \mathcal{S}^t \to \mathbb{R}$.

The term "quasi-planner" is justified by two assumptions. First, as said before, the quasi-planner cannot transfer resources across islands and, second, the quasi-planner is price-taker. The quasi-planner will indeed consider after-tax prices $(r_t : \mathcal{S}^t \to \mathbb{R})_{t\geq 0}$ and $(w_t : \mathcal{S}^t \to \mathbb{R})_{t\geq 0}$ as given.

Size of the islands.

Using the law of large numbers, we can deduce the size of the e^N-island from the transition matrix $M(s_t)$. Note first that a N-period idiosyncratic history $e^N \in E$ can be written as a sequence $e^N = (e_{N-1}, \ldots, e_0)$. We start with computing the probability $\Pi_{t,\hat{e}^N,e^N}$ –again following notation conventions from Remark 1– that an household having experienced history $\hat{e}^N$ in period t, experiences history e^N at period $t+1$. This probability can be expressed as the probability to switch from the state $\hat{e}_0$ at t to state e_0 at $t+1$, provided that histories $\hat{e}^N$ and e^N are compatible. More formally:

$$\Pi_{t,\hat{e}^N,e^N} = 1_{e^N \succeq \hat{e}^N} M_{\hat{e}_0,e_0}(s_t), \tag{8}$$

where $1_{e^N \succeq \hat{e}^N} = 1$ if e^N is a possible continuation of history $\hat{e}^N$. We now turn to the island size. At any date t, we denote as S_{t,e^N} the size of e^N-island, which is equal to the measure of households experiencing an individual $N-$period history e^N at period t. More formally, $S_{t,e^N} = \ell(i \in J | e^{i,t,N} = e^N)$, where we remind that the entire population is distributed on a segment J with measure ℓ. The measure S_{t,e^N} evolves according to the following dynamics, for $t \geq N$:

$$S_{t+1,e^N} = \sum_{\hat{e}^N \in \mathcal{E}^N} S_{t,\hat{e}^N} \Pi_{t,\hat{e}^N,e^N}. \tag{9}$$

Equation (9) simply states that agents on island e^N at date t come from island $\hat{e}^N \in \mathcal{E}^N$ at date $t-1$ with probability $\Pi_{t,\hat{e}^N,e^N}$.

To simplify the exposition of the model, we assume that in period 0, households enter the economy having already experienced a N-period history $e^N \in \mathcal{E}^N$. As a consequence, the measure $(S_{-1,e^N})_{e^N \in \mathcal{E}^N}$ is considered as given (with $\sum_{e^N \in \mathcal{E}^N} S_{-1,e^N} = 1$). Moreover, agents having the same initial N-period history e^N enter the economy in period 0 with the same initial wealth denoted $(a_{-1,e^N})_{e^N \in \mathcal{E}^N}$. The law of motion (9) is thus valid from period 0 onwards.[6]

Timing of events.

The timing of events is the following. Consider an agent with history $\hat{e}^N = (\hat{e}_{N-1}, \ldots, \hat{e}_0)$ at date $t-1$, living on $\hat{e}^N$-island at the end period $t-1$, with an end-of-period asset holding $a_{t-1,\hat{e}^N}$. The timing is as follows:

1. At the beginning of period t, she learns her new individual status e_t and aggregate risk realization z_t at date t. She is therefore endowed with the $N-$period history $e^N = (\hat{e}_{N-2}, \ldots, \hat{e}_0, e_t)$ (history is shifted by one period).

2. She moves to the e^N-island with her individual asset holding $a_{t-1,\hat{e}^N}$.and the period t per capita beginning-of-period resources of the e^N-island is equally shared among all agents of the island, who have pooled all their resources in common:

$$\tilde{a}_{t,e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S_{t-1,\tilde{e}^N}}{S_{t,e^N}} \Pi_{t-1,\tilde{e}^N,e^N} a_{t-1,\tilde{e}^N}. \tag{10}$$

As the initial level of wealth a_{-1,e^N} is given by initial conditions, equation (10) is true for

[6] As will become clear after the presentation of the economy, these assumptions are made without loss of generality.

$t \geq 0$ onwards.

3. The central planner decides the common consumption of island inhabitants c_{t,e^N} and their new individual asset holding a_{t,e^N}.

3.2 Program of the quasi-planner

We now provide a formulation for the program of the planner. The program of the quasi-planner consists in choosing consumption, saving and labor paths so as to maximize the expectation over all possible future states of the aggregate welfare on all islands. We denote $\mathbb{E}_0$ as the expectation operator at date 0 over all future aggregate histories. The program of the planner can be expressed as follows:

$$\max_{\left(a_{t,e^N},c_{t,e^N},l_{t,e^N}\right)_{t\geq 0,e^N\in\mathcal{E}^N}} \mathbb{E}_0\left[\sum_{e^N\in\mathcal{E}^N} S_{t,e^N}U\left(c_{t,e^N},l_{t,e^N}\right)\right] \tag{11}$$

$$a_{t,e^N}+c_{t,e^N}=w_t n_{e_t^N} l_{t,e^N}+\delta 1_{e_t^N=0}+(1+r_t)\tilde{a}_{t,e^N}, \text{ for all } e^N\in\mathcal{E}^N, \tag{12}$$

$$c_{t,e^N}, l_{t,e^N}\geq 0, a_{t+1,e^N}\geq -\bar{a}, \text{ for all } e^N\in\mathcal{E}^N, \tag{13}$$

$$S_{-1,e^N} \text{ and } a_{-1,e^N} \text{ are given.} \tag{14}$$

and subject the laws of motion for S_{t,e^N} given by (9), and to the definition of $\tilde{a}_{t,e^N}$ given by (10). Note that $1_{e_t^N=0}$ is an indicator function equal to 1 if island inhabitant are unemployed at date t, i.e. if $e_t^N=0$, and 0 otherwise.

Let us detail the program (11)–(14). The quasi-planner maximizes the aggregate welfare (11) subject to the the budget constraints (12) on all islands, the positivity and borrowing constraints (70), and the initial conditions (14). In the budget constraints (12) of islands, we use the after-tax real interest rate r_t and wage rate w_t, while the term $w_t n_{e_{0,t}^N} l_{t,e^N}$ is the per capita labor income on island e^N. This income is equal to the wage rate w_t multiplied by the productivity $n_{e_t^N}$ and the labor supply l_{t,e^N} on the island e^N. This labor income reduces to 0 on islands where agents are unemployed at date t, i;e., such that $e_t^N=0$. On these islands, where $e_t^N=0$ at date t, the income is equal to home production δ and is independent of labor effort. The equations (13) include non-negativity constraints for consumption and the credit constraint in each island. The existence of a recursive formulation of the problem is straightforward, as the problem is concave and the constraints (9) and to the and (10) are linear. Note ν_{t,e^N} as the Lagrange multiplier of the credit constraint $a_{t,e^N} \geq \bar{a}$ on island e^N. Solving this program yields, for all $e^N \in \mathcal{E}^N$:

$$U_c\left(c_{t,e^N}, l_{t,e^N}\right) + \nu_{t,e^N} = \beta \mathbb{E}_t \left[\sum_{\hat{e}^N \in \mathcal{E}^N} \Pi_{t,e^N,\hat{e}^N} U_c\left(c_{t+1,\hat{e}^N}, l_{t+1,\hat{e}^N}\right)(1+r_{t+1}) \right], \tag{15}$$

$$w_t n_{e_t^N} U_c\left(c_{t,e^N}, l_{t,e^N}\right) = -U_l\left(c_{t,e^N}, l_{t,e^N}\right) \text{ if } e_t^N > 0, \tag{16}$$

$$l_{t,e^N} = 0 \text{ if } e_t^N = 0. \tag{17}$$

$$\nu_{t,e^N}(a_{t,e^N} + \bar{a}) = 0 \text{ and } \nu_{t,e^N} \geq 0, \tag{18}$$

where $\mathbb{E}_t$ is the expectation operator conditional on the history of aggregate shocks up to date t. The Lagrange coefficient ν_{t,e^N} is zero when $a_{t,e^N} > -\bar{a}$. If $e_t^N = 0$, then agents are unemployed in the island e^N at date t. There is no labor supply $l_{t,e^N} = 0$, and the per capita income on this island is home production δ. We can further specify Euler equations by using the GHH utility specification (1) and obtain

$$u'\left(c_{t,e^N} - \frac{l_{t,e^N}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right) + \nu_{t,e^N} = \beta \mathbb{E}_t \left[\sum_{\hat{e}^N \in \mathcal{E}^N} \Pi_{t,e^N,\hat{e}^N} u'\left(c_{t+1,e^N} - \frac{l_{t+1,e^N}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right)(1+r_{t+1}) \right], \tag{19}$$

$$l_{t,e^N} = \left(w_t n_{e_t^N}\right)^{\varphi}. \tag{20}$$

Note that the individual labor supply in equation (20) is valid for any islands, including those where $e_t^N = 0$, although this expression originally stems from equation (16), which is valid on islands where $e_t^N > 0$.

3.3 Market clearing conditions and government budget constraint

Before turning to the sequential equilibrium definition, we provide the expression of aggregate quantities in the economy. To aggregate, we simply sum over all islands $e^N \in \mathcal{E}^N$ of size S_{t,e^N} the individual quantities. The total labor supply in efficient unit at date t amounts to

$$L_t = \sum_{e^N \in \mathcal{E}^N} n_{e_t^N} S_{t,e^N} l_{t,e^N}. \tag{21}$$

We can deduce from the above equation that the aggregate labor supply L_t only depends on the aggregate shock of date $t-1$. Indeed, plugging the law of motion of S of equation (9), equation (20) determining the individual labor supply l_{t,e^N} and equation (4) of the individual wage, we obtain

$$L_t = \sum_{e^N \in \mathcal{E}^N} n_{e_t^N} \sum_{\hat{e}^N \in \mathcal{E}^N} S_{t-1,\hat{e}^N} \Pi_{t-1,\hat{e}^N,e^N} \left(F_L(K_{t-1}, L_t, s_{t-1}) n_{e_t^N} \right)^{\varphi}, \tag{22}$$

which shows that L_t does not depend on the current aggregate shock. Together with equation (3), this enables us to state the following lemma:

Lemma 1 (Interest rate) *The interest rate on capital r_t is determined as date $t-1$.*

Lemma 1 confirms our choice of having the same interest rate for all interest-bearing assets

The end-of-period saving of all agents, denoted A_t at date t can be expressed as

$$A_t = \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} a_{t,e^N} = \sum_{e^N \in \mathcal{E}^N} S_{t+1,e^N} \tilde{a}_{t+1,e^N}, \tag{23}$$

where the last equality stems from equation (10) defining the relationship between $(\tilde{a}_{t+1,e^N})$ and (a_{t,e^N}). This last equality means that the transfer of wealth across islands does not affect the total amount of wealth. The clearing of the financial market at date t implies that at any date t, the following equality holds:

$$A_t = B_t + K_t. \tag{24}$$

The government budget constraint is given by equation (6). Factor prices are given by equations (3) and (4). Finally, from factor prices, one can recover the tax rates by the two equalities

$$\tau_t^K = 1 - \frac{r_t}{\tilde{r}_t} \text{ and } \tau_t^L = 1 - \frac{w_t}{\tilde{w}_t}. \tag{25}$$

3.4 Sequential equilibrium definition

We can provide the formal definition of our sequential equilibrium for the quasi-planner when the fiscal policy specified by taxes and public debt is given.

Definition 1 (Sequential equilibrium) *A sequential competitive equilibrium for the quasi-planner is a collection of individual allocations $\left(c_{t,e^N}, l_{t,e^N}, \tilde{a}_{t,e^N}, a_{t,e^N}\right)_{t\geq 0, e^N \in \mathcal{E}^N}$, of island population sizes $\left(S_{t,e^N}\right)_{t\geq 0, e^N \in \mathcal{E}^N}$, of aggregate quantities $(L_t, A_t, B_t, K_t)_{t\geq 0}$ and of price processes*

$(w_t, r_t, \tilde{r}_t, \tilde{w}_t)_{t\geq 0}$ *such that, for a given fiscal policy* $(\tau_t^K, \tau_t^L, B_t)_{t\geq 0}$*, for an initial distribution of island population and wealth* $\left(S_{-1,e^N}, a_{-1,e^N}\right)_{e^N\in\mathcal{E}^N}$*, for initial values of the capital stock* $K_{-1} = \sum_{e^N\in\mathcal{E}^N} S_{-1,e^N} a_{-1,e^N}$*, of the public debt* B_{-1} *and of the initial aggregate shock* s_{-1}*, we have:*

1. *given prices, individual strategies* $\left(a_{t,e^N}, c_{t,e^N}, l_{t,e^N}\right)_{t\geq 0, e^N\in\mathcal{E}^N}$ *solve the agents' optimization program in equations (11)–(14);*
2. *island sizes and beginning of period individual wealth* $\left(\tilde{a}_{t,e^N}, S_{t,e^N}\right)_{t\geq 0, e^N\in\mathcal{E}^N}$ *are consistent with law of motions (10) and (9);*
3. *labor and financial markets clear at all dates: for any* $t \geq 0$*, equations (21)–(24) hold; **
4. *the government budget constraint (6) holds at any date;*
5. *factor prices* $(w_t, r_t, \tilde{r}_t, \tilde{w}_t)_{t\geq 0}$ *are consistent with (3), (4) and the two equalities (25).*

The quasi-planner equilibrium is a finite-state equilibrium defined by $6(E+1)^N+8$ variables and $6(E+1)^N+8$ equations. The main interest of the setup described in this section is that we can provide conditions to show that it converges when N grows to a full-fledge incomplete insurance-market economy with aggregate shocks. To prove this property, the next section provides a decentralization of the previous allocation.

4 The decentralized economy

In this section, we explain how we can decentralize the allocation of the quasi-planner by choosing a proper transfer system. This decentralization will allow studying the properties of the convergence of the quasi-planner economy when N increases. This Section can be skipped if the reader is convinced by the relevance of the quasi-planner representation and wants to directly look at Ramsey policy in this environment.

In the decentralized economy, agents maximize their intertemporal welfare using both public debt shares and claims on the capital stock to self-insure against both idiosyncratic and aggregate risks. Financial markets are incomplete for both risks, because agents have just one asset (capital or public debt that pay the same interest rate) to self-insure. The decentralization relies on a particular additional transfer scheme. In every period, agents pay or receive a lump-sum transfer, which is contingent on their individual history over the previous N periods. In equilibrium, this transfer will ensure that agents sharing the same $N-$period history $e^N \in \mathcal{E}^N$ will have the same after-transfer beginning-of-period wealth. These transfer payments allow reducing the heterogeneity among agents, since they "pool" together all agents with the same history e^N. We now provide a detailed and formal construction of this decentralized equilibrium, where we will in particular show that decentralized Euler equations match those of the quasi-planner of Section 3.

4.1 The program of the agents

As in the quasi-planner economy, we construct a recursive equilibrium to simplify the exposition.[7] The economy is now populated by agents who are expected-utility maximizers. The value function of an agent will depend on three state variables: (i) the personal history of the realization of the previous $N-$period idiosyncratic shocks in addition to the current one, (ii) the beginning-of-period and before-tax wealth a and (iii) other (exogenous) state variables describing the state of the world and used by the agent to form her rational expectations, which we denote $\tilde{z}$. Let us be a bit more specific about personal history as a state variable. Consider an agent that enters the period with the $N-$period history $\tilde{e}^N \in \mathcal{E}^N$. Before making any decision, she learns the realization of her idiosyncratic shock $e \in \mathcal{E}$ for the current period. She is now endowed with the $N+1-$period history $e^{N+1} = (\tilde{e}^N, e) = (e^N_N, e^N_{N-1}, \ldots, e^N_1, e)$. This $N+1-$period history will drive the lump-sum transfer $T(e^{N+1}, \tilde{z})$ payment the agent has to make. To lighten the notations, we now omit the dependance of variables on $\tilde{z}$, when it does not create confusions. The agent maximizes her intertemporal welfare by choosing the current consumption c, labor effort l and asset holding a'. She will have to pay an after distorting-tax interest rate and wage rate denoted as r and w, as before. The value function can be written as

$$V(a, e^{N+1}) = \max_{a',c,l} U(c,l) + \beta \mathbb{E} \sum_{e' \in \mathcal{E}} M_{e,e'} V(a', (e^N_{N-1}, \ldots, e, e')) \tag{26}$$

$$a' + c + T(e^{N+1}) = w n_e l + \delta 1_{e=0} + (1+r)a \tag{27}$$

$$c, l \geq 0, a' \geq -\bar{a} \tag{28}$$

[7] The problem of the agents could be presented in a sequential form, at the cost of much heavier notations.

In the value function (26), the expectation operator $\mathbb{E}$ is taken with respect to the future aggregate shock $\tilde{z}'$ only. The expectation with respect to the idiosyncratic shock is written explicitly in the sum $\sum_{e' \in \mathcal{E}} M_{e,e'}$, where $M_{e,e'}$ is the probability to switch from the idiosyncratic state $e \in \mathcal{E}$ to $e' \in \mathcal{E}$. In this case, the next-period $N+1$ history will be $(e^N_{N-1}, \ldots, e, e')$ (which implies that e^{N+1} is shifted exactly by one period). Equation (27) is the budget constraint of the agent, where r and w denote the after-tax interest rate and wage rate. Agent's resources are made of asset payoffs $(1+r)a$ and of income earnings, which are $wn_e l$ when employed (and equals 0 when unemployed $e = 0$) and δ when not (i.e., $e = 0$). The agent use these resources to consume c, purchase assets a' and pay transfer $T(e^N)$. Positivity constraints for consumption and labor, as well as borrowing constraints are provided in equations (28).

We denote as $\tilde{\eta}(a, e^{N+1})$ the Lagrange coefficient of the credit constraint $a' \geq -\bar{a}$. The solution to the maximization program (26)–(28) are the policy rules denoted $c = g_c(a, e^{N+1})$, $a' = g_{a'}(a, e^{N+1})$, $l = g_l(a, e^{N+1})$ and the multiplier $\tilde{\eta}(a, e^{N+1})$ satisfying the following first order conditions:

$$U_c\left(c, l\right) + \tilde{\eta} = \beta \mathbb{E} \sum_{e' \in \mathcal{E}} M_{e,e'}\left(\tilde{z}\right) U_c\left(c', l'\right)\left(1 + r'\right), \tag{29}$$

$$U_l(c, l) + wn_e U_c(c, l) = 0, \text{ if } e \in \mathcal{E} - \{0\} \tag{30}$$

$$l = 0 \text{ if } e = 0, \tag{31}$$

$$\tilde{\eta}(a' + \bar{a}) = 0 \text{ and } \eta \geq 0. \tag{32}$$

4.2 Transfer scheme

We can now construct the transfer scheme following a guess-and-verify strategy. The transfer is constructed such that all agents with the same individual history over the last N periods will have the same after-transfer wealth. Consider agents with a beginning-of-period history $\tilde{e}^N \in \mathcal{E}^N$ who experience an idiosyncratic risk realization $e \in \mathcal{E}$ in the current period. The measure of agents with history $\tilde{e}^N$ is $S_{\tilde{e}^N}$, while the measure of agents with history $e^N \in \mathcal{E}^N$ is S'_{e^N}. The law of motion of S'_{e^N} is given by:

$$S'_{e^N} = \sum_{\hat{e}^N \in \mathcal{E}^N} S_{\hat{e}^N} \Pi_{\hat{e}^N, e^N}, \tag{33}$$

which is the parallel the law of motion (9) in the quasi-planner economy. We now assume that for any $\tilde{e}^N \in \mathcal{E}^N$, agents having the history $\tilde{e}^N$ have the same beginning-of-period wealth $a_{\tilde{e}^N}$. For agents having transited to the history e^N to hold the same asset quantity denoted $\tilde{a}(e^N)$, the following equality must hold:

$$\tilde{a}_{e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S_{\tilde{e}^N}}{S'_{e^N}} \Pi_{\tilde{e}^N, e^N} a_{\tilde{e}^N}, \text{ for all } e^N. \tag{34}$$

The transfer borne by an agent having the current $N+1$- period history $e^{N+1} = (\tilde{e}^N, e)$ has to mimic this pooling operation and is equal to

$$T_{e^{N+1}} = (1+r)\left(\tilde{a}_{e^N} - a_{\tilde{e}^N}\right) \tag{35}$$

The transfer T swaps the remuneration of the beginning of period wealth of agents having history $\tilde{e}^N$ by the remuneration of the average wealth $\tilde{a}_{e^N}$ of agents having the current $N-$period history $e^N = (\tilde{e}_{N-2}, \ldots, \tilde{e}_0, e)$. As a consequence, this transfer is balanced, as is shown by the next Lemma. The proof is in Appendix.

Lemma 2 (Balanced transfer scheme) *In each period t, we have*

$$\forall e^N \in \mathcal{E}^N,\ \sum_{\tilde{e} \in E} S'_{(\tilde{e}, e^N)} T_{(\tilde{e}, e^N)} = 0.$$

Agents are transfer-takers. All agents consider the lump-sum transfer T as given and thus do not internalize the effect of their choice on this transfer. Indeed, there is a continuum –with positive mass $S(\tilde{e}^N)$– of agents with history $\tilde{e}^N$ and each individual agent with history $\tilde{e}^N$ is atomistic in this set. As a consequence, an individual agent has no effect on the aggregate wealth of agents with history $\tilde{e}^N$ and therefore no effect on the transfer T.

The impact of transfer on agents' wealth. We consider the impact of transfer $T_{e^{N+1}}$ for an agent with history $e^{N+1} = (\tilde{e}^N, e) \in \mathcal{E}^{N+1}$ and beginning-of-period wealth $a_{\tilde{e}^N}$. Her budget constraint (27) can be expressed using transfer expression (35) as follows:

$$a' + c = w n_{e_0} l + \delta 1_{e_0=0} + (1+r)\tilde{a}_{e^N} \tag{36}$$

The beginning-of-period and after-transfer wealth of agents with history e^{N+1} only depends on the current $N-$period history e^N. Moreover, as can be seen from (26), agents with the same $N-$period history e^N are endowed with the same expected continuation utility if they save the same amount a'. Therefore, agents with the same current $N-$period history e^N behave similarly: they consume the same level, they supply the same labor quantity and hold the same wealth.

As a consequence, the solution to the maximization program (26)–(28) are policy rules $c = g_c(a, e^N)$, $a' = g_{a'}(a, e^N)$, $l = g_l(a, e^N)$ and the multiplier $\tilde{\eta}(a, e^N)$.

4.3 Comparison with the quasi-planner economy

The key outcome of the construction of this truncated economy is that this economy maps the quasi-planner environment. To see this, one can first write the Euler equations (29)–(32) defining consumption level, labor supply and borrowing constraints in sequential terms. Using the relationship (8), one can observe that they are the same as (15)–(18). Second, the definitions of $\tilde{a}_{e^N}$ in equation s (34) and the budget constraints given by (12) and (36) are identical for both economies, when written in sequential terms.

4.4 Convergence of the transfer

The previous economy proves that the quasi-planner economy is consistent with the representation of Krusell and Smith (1998), which represent recursively a incomplete-market economy where the wealth distribution enters as a state variable. Nevertheless, the existence of the transfer $T^{(N)}_{e^{N+1}}$ in the decentralized economy may generate a systematic difference in terms of policy rules between the quasi-planner economy and the full-fledged economy where this transfer would be 0. Note that we now add a superscript (N) to the transfer notation to make the dependence in N explicit. In this Section, we exhibit some conditions under which this transfer tends toward 0, when N becomes infinitely large. We first state the Proposition and explain how it can be used.

Proposition 1 (Convergence of the transfer) *If there exist* $\Gamma \in (0,1)$ *and* $\overline{N} \geq 1$*, such that for all* $N \geq \overline{N}$ *and all* $\tilde{z}$*, we have for all* $(e_{\bar{N}-1}, \ldots, e_0) \in \mathcal{E}^{\overline{N}}$, $(\tilde{e}_{N-1}, \ldots, \tilde{e}_{\bar{N}}) \in \mathcal{E}^{N-\overline{N}}$, $(\hat{e}_{N-1}, \ldots, \hat{e}_{\bar{N}}) \in$

$\mathcal{E}^{N-\overline{N}}$:

$$\left|g^N(a', (\tilde{e}_{N-1}, \dots, \tilde{e}_{\bar{N}}, e_{\bar{N}-1}, \dots, e_0), \tilde{z}) - g^N(a, (\hat{e}_{N-1}, \dots, \hat{e}_{\bar{N}}, e_{\bar{N}-1}, \dots, e_0), \tilde{z})\right| < \Gamma\,|a' - a| \tag{37}$$

then

$$\lim_{N\to\infty} |T_{\cdot}^{(N)}| = 0.$$

The condition (37) is involved, but it has a simple meaning. It states that if we increase the beginning-of-period wealth of agents having the same history e^N (for $N \geq \overline{N}$) by an amount $\triangle a$, then these agents will save less than $\Gamma \triangle a$ ($\Gamma < 1$) in all states of the world: the increase in savings will be strictly smaller than the income increase. In other words, the saving propensity is strictly lower than 1. If this condition is fulfilled then, as the length of idiosyncratic history N increases, the transfer made to agents tends toward 0. The reason is the following. If the saving propensity is strictly lower than one, initial differences in wealth vanish and agents experiencing the same history of idiosyncratic shocks end up having a the same wealth. As a consequence, the pooling of wealth (34) concerns wealth levels which tends to be arbitrarily close to each other. As a consequence, the transfer tends toward 0 for large N. The proof is a little bit involved and can be found in Appendix.

Proposition 1 states conditions on policy rules and not on deep parameters of the model. It can nevertheless be used in two ways. The first one is to derive the set of policy rules as N grows to check that this condition is fulfilled. This may be very demanding as the size of this set increases rapidly. A second direction is to solve the model without aggregate shock, when the transfer is set to 0. In this case, it is known that a recursive formulation exists and one can find a very accurate approximation of policy rules (Huggett 1993; Aiyagari 1994). Then, if the policy rules exhibit a saving propensity that is always strictly lower than 1, then a continuity argument can be invoked to conclude that it will still hold if aggregate shocks are small enough. In this paper, we do not follow further this computational route. Instead, we show that this representation is useful to investigate optimal policies with incomplete-insurance markets and aggregate shocks in a relevant environment.

5 Ramsey problem

5.1 Problem formulation

We now turn to the resolution of Ramsey program in this incomplete market economy with aggregate shocks, where we take advantage of our limited heterogeneity equilibrium to solve the Ramsey program. The Ramsey program consists for the government to choose at date 0 a sequence of taxes on capital and labor as well as a path of public debt level that maximizes the aggregate welfare of the economy, assuming that individual agents behave rationally and subject to constraints on the economy-wide resources and on the government budget. In other words, it consists in finding the paths for taxes and public debt that selects the competitive equilibrium associated to the largest aggregate welfare. The following definition formalizes this statement.

Definition 2 (Ramsey program for a truncated economy) *Let $N > 0$. Given initial conditions about the wealth distribution $\left(S_{-1,e^N}, a_{-1,e^N}\right)_{e^N \in \mathcal{E}^N}$, the initial public debt B_{-1} and the initial state aggregate state s_{-1}, the Ramsey program consists in choosing, at date 0, a fiscal policy made of capital and labor tax paths $(\tau_t^K, \tau_t^L)_{t\geq 0}$ and of public debt paths $(B_t)_{t\geq 0}$, that maximizes the aggregate welfare among the set of competitive equilibria characterized in Definition 1.*

In other words, the Ramsey program consists in setting taxes and public debt at $t = 0$, such that the government maximizes the aggregate welfare while internalizing individual agents objectives and constraints.

Following our result that shows that the quasi-planner equilibrium coincides with the truncated economy, the Ramsey program can be expressed as finding paths for taxes and public debt for maximizing the quasi-planner program subject to resource constraints and government budget constraints. Moreover, given the expression of distorting taxes τ_t^K and τ_t^L in equation (25) , it is equivalent for the government to decide the post-tax interest rate $(r_t)_{t\geq 0}$ and the post-tax wage rate $(w_t)_{t\geq 0}$ instead of distorting taxes. In consequence, we can formalize the

Ramsey program as follows:

$$\max_{\left(r_t, w_t, B_t, (a_{t,e^N}, c_{t,e^N}, l_{t,e^N})_{e^N \in \mathcal{E}^N}\right)_{t \geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} U(c_{t,e^N}, l_{t,e^N}) \right], \tag{38}$$

$$B_t + F(K_{t-1}, L_t, s_t) = G_t + (1 + r_t) B_{t-1} + r_t K_t + w_t L_t, \tag{39}$$

$$a_{t,e^N} + c_{t,e^N} = w_t n_{e^N} l_{t,e^N} + \delta 1_{e_t^N = 0} + (1 + r_t) \tilde{a}_{t,e^N}, \text{ for all } e^N, \tag{40}$$

$$U_c \left(c_{t,e^N}, l_{t,e^N} \right) + \nu_{t,e^N} = \beta \mathbb{E}_t \left[\sum_{\hat{e}^N \in \mathcal{E}^N} \Pi_{t+1,e^N,\hat{e}^N} U_c \left(c_{t+1,e^N}, l_{t+1,e^N} \right) 1 + r_{t+1}) \right] \tag{41}$$

$$l_{t,e^N} = \left(w_t n_{e_t^N} \right)^{\varphi} \text{ if } e_t^N > 0 \tag{42}$$

$$l_{t,e^N} = 0 \text{ if } e_t^N = 0. \tag{43}$$

$$\nu_{t,e^N} (a_{t,e^N} + \overline{a}) = 0, \tag{44}$$

$$\tilde{a}_{t,e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S_{t-1,\tilde{e}^N}}{S_{t,e^N}} \Pi_{t-1,\tilde{e}^N,e^N} a_{t-1,\tilde{e}^N}, \tag{45}$$

$$S_{t,e^N} = \sum_{\tilde{e}^N \in \mathcal{E}^N} S_{t-1,\tilde{e}^N} \Pi_{t-1,\tilde{e}^N,e^N}, \tag{46}$$

$$A_t = \sum_{e \in \mathcal{E}^N} S_{t,e^N} a_{t,e^N}, \tag{47}$$

$$L_t = \sum_{e \in \mathcal{E}^N} S_{t,e^N} l_{t,e^N}, \tag{48}$$

$$K_t = A_t + B_t, \tag{49}$$

$$c_{t,e^N}, l_{t,e^N}, (a_{t,e^N} + \overline{a}) \geq 0. \tag{50}$$

where all constraints (39)–(50) should be understood, unless specified, for all $s^t \in \mathcal{S}^t$ and all $e^N \in \mathcal{E}^N$. Consistently with previous notations, the operator $\mathbb{E}_t$ is the conditional expectation at date t.

Let us comment the Ramsey program (38)–(50). Equation (38) is the objective and corresponds to the maximization of the aggregate welfare with an additive criterion. Maximization devices are on one hand individual quantities of consumption, labor and asset holdings and one the other hand the fiscal instruments, which are public debt and post-tax interest and wage rates. Equation (39) is the government budget constraint, while the individual budget constraint is given in equation (40). The individual Euler equations for consumption and labor are provided in equations (41) and (42) respectively. The multiplier ν_{t,e^N} appears in the slackness condition (44), stating that either the borrowing constraint multiplier ν_{t,e^N} on island e^N is null, or the borrowing constraint for island e^N binds: $a_{t,e^N} = -\overline{a}$. Equation (45) defines the pooling operation that occurs when agents transit from islands to others, while the dynamics of island sizes is specified in equation (46). Equations (47) and (48) provide the aggregation for individual wealth and labor supply, while the financial market clearing is given by equation (49). Finally, positivity and borrowing constraints appear in equation (50).

5.2 Simplification of the Ramsey program

In this section, we simplify the formulation of the Ramsey program exposed in equations (38)–(50). For sake of clarity, we drop in the remainder the "accounting" equalities (46)–(49) and positivity constraints (50). We first denote $\beta^t m^t(s^t) S_{t,e^N} \lambda_{t,e^N}$ the (normalized) Lagrange multiplier of the Euler equation of agent e^N in state s^t. We now define for all $e^N \in \mathcal{E}^N$:

$$\Lambda_{t-1,e^N} \equiv \frac{\sum_{\hat{e}^N \in \mathcal{E}^N} S_{t-1,\hat{e}^N} \lambda_{t-1,\hat{e}^N} \Pi_{t,e^N,\hat{e}^N}}{S_{t,e^N}}, \tag{51}$$

which can be interpreted as an average previous-period Lagrange multiplier of agents having period -t history e^N, where the quantities $\lambda_{t,\hat{e}^N}$ for any $\hat{e}^N \in \mathcal{E}^N$ are weighted the size of population transiting from state $\hat{e}^N$ to e^N. We also normalize $\lambda_{-1} = 0$ for sake of simplicity (and without loss of generality). Finally, we can notice that we have $\lambda_{t,e^N} = 0$ if $a_{t,e^N} = 0$. We can therefore drop the product of Lagrange multipliers $\lambda_{t,e^N} \nu_{t,e^N}$ (for any t and any e^N) that are in fact redundant. We also substitute the labor expressions (42) and (43) in the Ramsey program. Note that since $e_t^N = 0$ is equivalent to $n_{e_t^N} = 0$ and since $\varphi > 0$, equality (43) is implied by (42).

The following lemma summarizes our simplification of the Ramsey program.

Lemma 3 (Simplified Ramsey program) *The Ramsey program in equations (68)–(50) can* *be simplified into:*

$$\max_{\left(r_t, w_t, B_{t+1}, (a_{t+1,e^N}, c_{t,e^N})_{e^N\in\mathcal{E}^N}\right)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N\in\mathcal{E}^N} S_{t,e^N} \left(U(c_{t,e^N}, \left(w_t n_{e_t^N}\right)^{\varphi}) \right. \tag{52}$$

$$\left. + U_c(c_{t,e^N}, \left(w_t n_{e_t^N}\right)^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N}\right)\right)$$

$$s.t.\ \lambda_{t,e^N} = 0 \ \textit{if}\ a_{t,e^N} = 0, \tag{53}$$

$$B_t + F(A_{t-1} - B_{t-1}, L_t, s_t) = G_t + (1+r_t)B_{t-1} + r_t(A_{t-1} - B_{t-1}) + w_t L_t, \tag{54}$$

$$a_{t,e^N} + c_{t,e^N} = \left(w_t n_{e_t^N}\right)^{\varphi+1} 1_{e_t^N>0} + \delta 1_{e_t^N=0} \tag{55}$$

$$+ (1+r_t) \sum_{\hat{e}\in\mathcal{E}^N} \frac{S_{t-1,\hat{e}^N}}{S_{t,e^N}} \Pi_{t,\hat{e}^N,e^N} a_{t-1,\hat{e}^N}$$

The proof is relegated in Appendix.

5.3 Ramsey program first-order conditions

We finally compute the first order conditions associated to the Ramsey program (52)–(55). The following proposition summarizes our findings:

Proposition 2 (Ramsey program first-order conditions) *First order conditions associated to the Ramsey program can be expressed as follows*

$$U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N}\right) + \beta \mathbb{E}_t \left[\mu_{t+1}(r_{t+1} - F_K(A_t - B_t, L_t))\right] \tag{56}$$

$$= \beta \mathbb{E}_t \sum_{\tilde{e}^N\in\mathcal{E}^N} (1+r_{t+1}) \Pi_{t+1,e^N,\tilde{e}^N}$$

$$\times \left(U_{cc}(c_{t+1,e^N}, (w_{t+1} n_{e_{t+1}^N})^{\varphi}) \left(\Lambda_{t,e^N}(1+r_{t+1}) - \lambda_{t+1,e^N}\right)\right)$$

$$\mu_t = \beta \mathbb{E}_t \left[\mu_{t+1}\left(F_K(A_t - B_t, L_{t+1}, s_t) + 1\right)\right]. \tag{57}$$

$$\mu_t A_{t-1} = \sum_{e\in\mathcal{E}^N} S_{t,e^N} U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \Lambda_{t-1,e^N} \tag{58}$$

$$+ \sum_{e\in\mathcal{E}^N} \left(\sum_{\hat{e}^N\in\mathcal{E}^N} S_{t-1,\hat{e}^N} \Pi_{t,\hat{e}^N,e^N} a_{t-1,\hat{e}^N}\right) \psi_{t,e_t^N}(c_{t,e^N})$$

$$\mu_t \left(L_t + \varphi L_t - \frac{\varphi}{w_t} F_L(K_{t-1}, L_t) L_t\right) = (1-\varphi) \sum_{e^N\in\mathcal{E}^N} S_{t,e^N} n_{e_t^N} (w_t n_{e_t^N})^{\varphi} \psi_{t,e_t^N}(c_{t,e^N}) \tag{59}$$

where

$$\psi_{t,e_t^N}(c_{t,e^N}) = U_c(c_{t,e^N},(w_t n_{e_t^N})^{\varphi}) + U_{cc}(c_{t,e^N},(w_t n_{e_t^N})^{\varphi})\left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N}\right) \qquad (60)$$

Proof. The proof is relegated in Appendix. ■

Before interpreting further this set of equations, let us start with commenting the definition of ψ_{t,e_t^N} in equation (60). The quantity ψ_{t,e_t^N} measures the marginal impact of a change in consumption at date t for agent e^N when the agent's Euler equation is internalized by the planner. This quantity can be seen as the sum of two terms. The first term is a direct effect equal to the agent's consumption marginal utility. The second term is an indirect term reflecting the impact of a change in consumption on the individual Euler equation. Equation (56) is the *modified* Euler equation. It equalizes the marginal cost of saving more today to the marginal benefit of $(1+r_{t+1})$ consumption units in the next period. Note the absence of direct effect in consumption (no marginal utilities U_c) in the equation, simply because the individual consumption Euler equation (41) still holds and direct effect terms cancel out. In this modified Euler equation, today's cost is the (internalized) saving price channel (through the term in U_{cc}). The expected benefit in the next period reflects the price channel again (term in U_{cc}) as well as the impact on the government budget constraint (term in μ_{t+1}): larger asset supply means higher capital and thus larger interest payments on one side but also larger production on the other side.

Equation (57) is the Euler equation for the multiplier of the government budget constraint. This equation sets equal the marginal benefit of a larger debt today to the expected cost of the next period. The benefit is a more slack government budget constraint today because a larger debt issuance, while the cost is the larger debt repayment. This equation appears in similar fashion in absence of aggregate risk in Acikgoz (2013).

Equation (58) equalizes the marginal costs and benefits of raising the after-tax interest rate or equivalently the tax on capital. The benefit, equal to$\mu_t A_{t-1}$, is simply a more slack government budget constraint and the benefit is all the larger when the interest-bearing asset supply is large (since it is the tax base). The cost is the sum of two terms. The first effect, proportional to $(\Lambda_{t-1,e^N})_{e^N}$ reflects the impact of a change in interest rate on the individual consumption Euler equation (41). The second term, which is proportional to individual asset holdings $(a_{t-1,e^N})_{e^N}$, is the direct impact on individual budget constraints through a change in the individual tax base.

Equation (59) is similar to equation (58) for the after-tax wage or equivalently the tax on wages. The benefit, proportional to the Lagrange multiplier μ_t, is the impact of the tax on the government budget constraint. Note that the direct positive effect (proportional to the labor supply L_t) is dampened by a distortion on the labor supply (terms in φL_t), since raising taxes affects the tax base. The cost consists in the direct impact on the individual budget constraints (terms in $n_{e_t^N}(w_t n_{e_t^N})^{\varphi}$). The two last terms reflect the distortion on individual labor decision.

6 A numerical application

We now provide a numerical application to investigate the properties of optimal public debt and taxes. Following the literature, we identify the uninsurable risk to the unemployment risk. We thus consider two idiosyncratic states corresponding to employment and unemployment. As there are two types of agents, we denote by e the variable referring to employed agents, and by u the variable referring to unemployed agents. Unemployed agents get the home production δ, while employed agents have a labor productivity $n(e)$. To uncover the trade-offs, we consider the simple case where there are two islands, $N = 1$. As a consequence, all employed agents will consume and save the same amount and all unemployed agents will consume and save the amount. The simple environment already exhibits a rich and non obvious time-varying optimal policy. In addition, the analysis of this equilibrium makes it explicit how perturbation method can be used to numerically solve for this equilibrium.

6.1 Assumptions and model equations

The utility function is $u(c, l) = \log(c - \frac{l^{1+1/\varphi}}{1+1/\varphi})$. We denote $c_{t,e}$ and $c_{t,u}$ the consumption of employed and unemployed agents, while $a_{t,e}$ and $a_{t,u}$ denote asset holdings. We construct our equilibrium with a guess-and-verify strategy. We assume that unemployed agents are always credit-constrained, such that $a_{t+1,u} = \overline{a} = 0$. We check that it is indeed the case, by showing that this property holds in steady sate as well as in transitory dynamics. This will notably require the aggregate shock to be not too volatile. This assumption also implies that $\tilde{a}_{t,e} = \alpha_{t-1} S_{t-1,e} a_{t,e} / S_{t,e}$ and $\tilde{a}_{t,u} = (1 - \alpha_{t-1}) S_{t-1,e} a_{t,e} / S_{t,u}$.

In this economy we consider three shocks. The first one is a shock on the idiosyncratic risk. We introduce a shock on the transition α_t such that the unemployment rate temporarily increases. This shock will help to identify optimal fiscal policy after a increase in uncertainty at the agents level. The second one shock is a technology shock affecting the production function. The third shock is public spending shock. We want to investigate the shape of the fiscal multiplier when optimal fiscal policy is implemented. The state is thus $s_t = (s_t^\alpha, s_t^a, s_t^g)$.

Uncertainty shock. The first exogenous process affects transition probabilities. Assume that

$$s_t^\alpha = (1 - \varphi_\alpha)\bar{\alpha} + \varphi_\alpha s_{t-1}^\alpha + \epsilon_t^\alpha \tag{61}$$

and the transitions are

$$\alpha_t = s_t^\alpha \text{and } \rho_t = \bar{\rho} \tag{62}$$

Technology shock. Technology is Cobb-Douglas with the following production function $F(K, L, s) = \Psi(s)K^\kappa L^{1-\kappa}$., where $\Psi(s) = e^s$. where z follows

$$s_t^a = \varphi_a s_t^a + \epsilon_t^a \text{ and } \Psi_t = e^{s_{t-1}^a} \tag{63}$$

Public spending shock. The process for public spending is assumed to be auto-regressive:

$$s_t^g = (1 - \varphi_g)\bar{G} + \varphi_g s_{t-1}^g + \epsilon_t^g \text{ and } G_t \tag{64}$$

The steady state is defined as an equilibrium where $\epsilon_t^\alpha = \epsilon_t^g = \epsilon_t^a = 0$ and where all real variables are constant.

The simple model and the equilibrium definition are detailed in Appendix E. It is basically the application of Proposition 2 to the case $N = 1$.

6.2 Parametrization

The period is a year. There are 6 parameters in the model. The subjective discount factor is set $\beta = 0.96$, to obtain a real interest close to 4%. The estimation of the elasticity of the labor supplies varies in the literature. We use $\varphi = 1$, and we check that the dynamics below don't

$\tilde{r}$	τ^K	τ^L	B/Y	λ_e	μ
4.17	14.7%	9.4%	17%	0.011	11.8

Table 1: Steady-state outcome of the model

change for alternative values $\varphi \in (0.3, 2.0)$, what is a standard range.

The capital share in production is $\kappa = 0.36$. The steady state value for public spending $\bar{G} = 0.063$ is set such that public spending in GDP is 40%, which is the value for the US in 2010. Concerning labor market transitions, the values are set to reproduce the annualized transitions in the US job market (Challe and Ragot, 2014). It implies $\bar{\alpha} = 0.95$ and $\bar{\rho} = 0.05$, i.e. a steady-state unemployment rate equal to 5%. As a benchmark, we set home production to $\delta = 0$. It implies a fall in consumption for agents switching from employment to unemployment of 14%. This last value is the direct measure of the lack of insurance in our model. This figure is well inside the range of available estimates (which range between 7 and 30 percent, see, e.g., Cochrane, 1991; Gruber, 1997; Chodorow-Reich and Karabarbounis 2015).

The steady state outcomes of the model are summed up in the following table.

The steady-state (before tax) real interest rate is 4.17%. The tax on capital is 14.7% and the steady-state tax on labor is 9.4%. Optimal debt over GDP is 17%. These fiscal values are much lower than the US average, which is 27% for tax on labor and 40% for the tax on capital. This low value of public debt over GDP was attained in 1930.

Finally, we set persistence parameters to $\varphi_\alpha = \varphi_a = \varphi_g = 0.9$ for the three shocks.

6.3 Fiscal policy after an uncertainty shock

The following graph plots the path of the real variables after a negative shock to α.

We denote dx for the percentage deviation of the variable x from its steady-state value. For example, *dalpha* is the percentage change in α. Both α and employment n falls from 1% on impact and goes back progressively to their steady state value. This generates a fall in output Y and capital K of roughly 1% on impact. The three policy variables, public debt and tax rate on capital and labor are then plotted. Public debt decreases on impact, whereas capital tax increases sharply for a small period. The labor tax increases persistently for much smaller amount. After such a policy, consumption of employed agents falls by less than 1.8% whereas the consumption of unemployed agents falls by 2.8% .

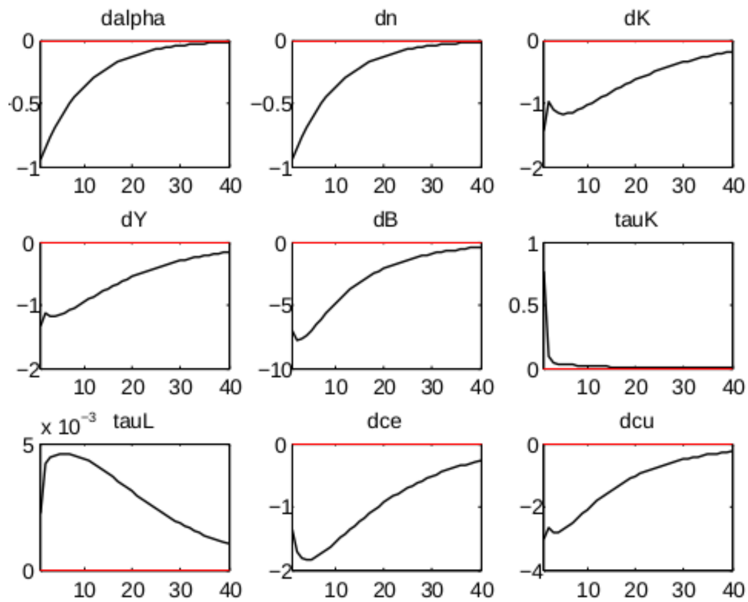

Figure 1: Impulse responses after a negative shock on α

What the the effect of this optimal fiscal policy? As capital is a fixed factor during the period, the sharp increase in capital tax is a way to get some resources to induce a fall in public debt, by 7%. This fall reduces the "crowding-out" effect of public debt and reduces the fall in capital. To see this, we simulated the economy fixing the capital tax to its steady state value, and optimizing only on labor tax. In this case, the reduction in public debt is only 3.8% and capital and output falls by 1.7% (instead of 1%). In this alternative economy, the consumption of employed households falls by 2.9%. After a negative shock on employment, optimal fiscal policy reduces the fall in capital by finding resources to reduce public debt.

6.4 Fiscal policy after a negative technology shock

The following graph plots key variables after a negative technology shock.

The fall of TFP of 1% generates a fall in capital and output of 3.1%. Note that as there is no change in the transitions, employment does not change, but labor supply falls on impact. As before, capital tax increases a lot on impact because capital is a fixed factor at the beginning of the period. This allows a sharp reduction in public debt of roughly 20% (debt over GDP falls from 17% to 13.5% of debt over steady-state GDP), what reduces the fall in capital after the negative technology shock. The consumption of employed agents falls by 5%, whereas the

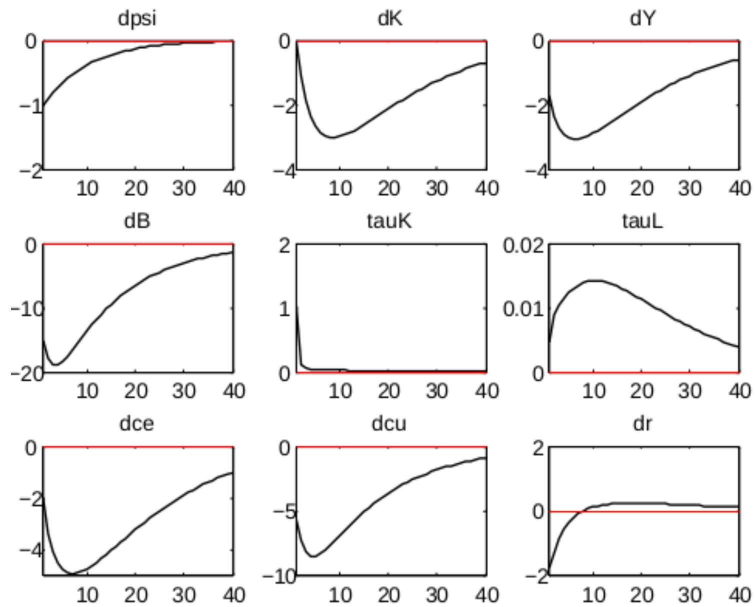

Figure 2: Impulse responses after a negative technology shock

consumption of unemployed agents falls by 8%. As before, keeping the capital tax equal to its steady-state value, one finds a much smaller fall in public debt (14%), and a larger fall in output, equal to 3.9%.

6.5 Fiscal policy after consumption spending shock

Finally, the last graph plots the effect of an increase of 1% in public spending financed by distorting taxes.

An increase in public spending increases distorting taxes and thus reduces output by 0.5%. As before the sharp increase in capital taxes allows for reducing public debt. This, in turn limits the fall in capital. Keeping distorting taxes on capital fixed, one finds a fall in output and consumption twice as big.

6.6 Concluding remarks on optimal fiscal policy

From this experience, we can conclude that optimal monetary policy after a adverse shock aims at containing the fall of capital by reducing public debt. This is obtained by an initial increase in capital taxes. Additionally, one finds that labor taxes are very smooth whereas capital taxes are very volatile. This result confirms the one found in Aiyagari, Marcet, Sargent and

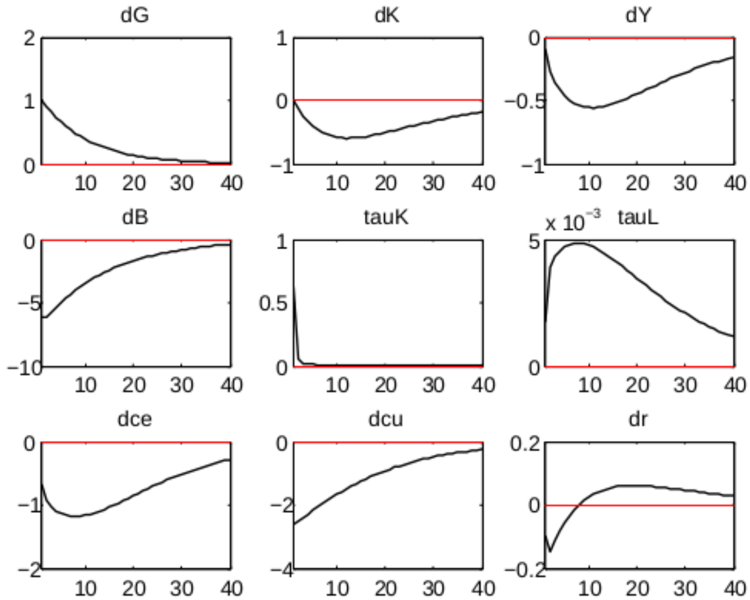

Figure 3: Impulse responses after a negative public spending shock

Seppala (2002), Farhi (2012) who considers an economy with a representative agent. Considering incomplete insurance market does not alter this general result.

Incomplete insurance markets generate a well defined optimal long-run level of public debt, which is used as a store of value by private agents. As a consequence, both taxes and public debt are mean-reverting after a transitory shock. In particular, public debt does not exhibit a unit root component as in Aiyagari, Marcet, Sargent and Seppala (2002). As capital and public debt compete as self-insurance device in a incomplete insurance-market economy, one finds that public debt falls after a negative shock to limit its "crowding out" effect. To our knowledge, this is a new finding about optimal debt management in the business cycle.

7 Concluding remarks

We have proved that the competitive equilibrium in an incomplete insurance market economy with aggregate shocks can be represented as the allocation of quasi-planner using limited fiscal instruments. The gain of this representation is that it generates a finite state-space equilibrium, in which the Ramsey outcome can be studied for various types of instruments and various types of aggregate shocks. This opens the possibility to study optimal policies in these environments.

We use this framework to study optimal fiscal policy, when distorting taxes on capital and labor and public debt are available. Compared to representative agent economy, the main distinguishing feature of the incomplete insurance-market economy, is that the optimal level of public debt is well defined, as agents hold both public debt and capital for self-insurance motives against idiosyncratic shocks. We find that all taxes and public debt are mean reverting, and that public debt decreases after a transitory negative shock, to reduce the fall in the capital stock.

References

Acikgoz, O. (2013): "Transitional Dynamics and Long-run Optimal Taxation Under Incomplete Markets," *MPRA Paper*, 50160.

Aiyagari, R. (1995): "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Journal of Political Economy*, 103(6), 1158–1175.

Aiyagari, R., A. Marcet, T. Sargent, and J. Seppala (2002): "Taxation without State Contingent Debt," *Journal of Political Economy*, 110(6), 1220–1254.

Aiyagari, R. S. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109(3), 659–684.

Aiyagari, R. S., and E. R. McGrattan (1998): "The Optimum Quantity of Debt," *Journal of Monetary Economics*, 42(3), 447–469.

Barro, R. (1979): "On the determination of the public debt," *Journal of Political Economy*, 87(5), 940–971.

Bewley, T. F. (1983): "A Difficulty with the Optimum Quantity of Money," *Econometrica*, 54(5), 1485–1504.

Bhandari, A., D. Evans, M. Golosov, and T. Sargent (2013): "Taxes, debts, and redistribution with aggregate shocks," *NBER Working paper*, 19470.

Challe, E., F. LeGrand, and X. Ragot (2013): "Incomplete Markets, Liquidation Risk and the Term Structure of Interest Rates," *Journal of Economic Theory*, 148(6), 2483–2519.

Challe, E., and X. Ragot (2014): "Precautionary Saving over the Business Cycle," *The Economic Journal*, forthcoming.

Chodorow-Reich, G., and L. Karabarbounis (2015): "The Cyclicality of the Opportunity Cost of Employment," *Journal of Political Economy*, Forthcoming.

Cochrane, J. H. (1991): "Production-Based Asset Pricing and the Link Between Stock Returns and Economic Fluctuations," *The Journal of Finance*, 46(1), 209–237.

Constantinides, G. M., and D. Duffie (1996): "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104(2), 219–240.

Dang, T. V., G. Gorton, B. Holmstrom, and G. Ordoñez (2014): "Banks as Secret Keepers," Working Paper 20255, NBER.

Farhi, E. (2012): "Capital Taxation and Ownership When Markets Are Incomplete," *Journal of Political Economy*, 118(5), 908–948.

Feldman, M., and C. Gilles (1985): "An Expository Note on Individual Risk without Aggregate Uncertainty," *Journal of Economic Theory*, 35(1), 26–32.

Green, E. (1994): "Individual-Level Randomness in a Nonatomic Population," Working paper, University of Minnesota.

Gruber, J. (1997): "The Consumption Smoothing Benefits of Unemployment Insurance," *The American Economic Review*, 87(1), 192–205.

Heathcote, J. (2005): "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *The Review of Economic Studies*, 72(1), 161–188.

Heathcote, J., K. Storesletten, and G. Violante (2014): "Optimal tax progressivity: An analytical Framework," *Working Paper*.

Huggett, M. (1993): "The Risk Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.

Judd, K. L. (1985): "The Law of Large Numbers with a Continuum of IID Random Variables," *Journal of Economic Theory*, 35(1), 19–25.

Kiyotaki, N., and J. Moore (2005): "2002 Lawrence R. Klein Lecture: Liquidity And Asset Prices," *International Economic Review*, 46(2), 317–349.

——— (2008): "Liquidity, Business Cycles and Monetary Policy," Working paper, Princeton University.

Krusell, P., T. Mukoyama, and A. A. J. Smith (2011): "Asset Prices in a Huggett Economy," *Journal of Economic Theory*, 146(3), 812–844.

Krusell, P., and A. A. J. Smith (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.

Lagos, R., and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113(3), 463–484.

Le Grand, F., and X. Ragot (2015): "Incomplete Markets and Derivative Assets," *Economic Theory*, Forthcoming.

Lucas, R., and N. Stockey (1983): "Optimal fiscal and monetary policy in an economy without capital," *Journal of Monetary Economics*, 12(1), 55–93.

Miao, J. (2006): "Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks," *Journal of Economic Theory*, 128(1), 274–298.

Miao, J., and P. Wang (2015): "Bubbles and Credit Constraints," Working paper, Boston University.

Uhlig, H. (1996): "A Law of Large Numbers for Large Economies," *Economic Theory*, 8(1), 41–50.

Appendix

A Proof of Lemma 2

From the definition of $\tilde{a}_{e^N}$ in equation (34), we obtain for any $e^N = (e^N_{N-1}, \ldots, e^N_1, e^N_0) \in \mathcal{E}^N$:

$$\begin{aligned} S'_{e^N}\tilde{a}_{e^N} &= \sum_{\hat{e}^N \in \mathcal{E}^N} S_{\hat{e}^N}\Pi_{\hat{e}^N, e^N} a_{\hat{e}^N} \\ &= \sum_{\hat{e} \in E} S_{(\hat{e}, e^N_{N-1}, \ldots, e^N_1)} M_{e^N_1, e^N_0}(\tilde{z}) a_{(\hat{e}, e^N_{N-1}, \ldots, e^N_1)}. \end{aligned}$$

Therefore, for a given $N-$period history $e^N \in \mathcal{E}^N$, we obtain (noting that $\sum_{\tilde{e} \in E} S'_{(\tilde{e}, e^N)} = S'_{e^N}$):

$$\begin{aligned} \sum_{\tilde{e} \in E} S'_{(\tilde{e}, e^N)} T_{(\tilde{e}, e^N)} &= (1+r)\left[\sum_{\tilde{e} \in E} S'_{(\tilde{e}, e^N)} \left(\tilde{a}_{e^N} - a_{(\tilde{e}, e^N_{N-1}, \ldots, e^N_1)}\right)\right] \\ &= (1+r)\left[S'_{e^N}\tilde{a}_{e^N} - \sum_{\tilde{e} \in E} S'_{(\tilde{e}, e^N)} a_{(\tilde{e}, e^N_{N-1}, \ldots, e^N_1)}\right] \\ &= 0, \end{aligned}$$

which concludes the proof.

B Proof of Proposition 1

We denote by $Conv(A)$ is the convex hull of the set A. We start with the following lemma.

Lemma 4 (contraction lemma) *If, for* $(e_{\bar{N}-1}, \ldots, e_0) \in \mathcal{E}^N, (f_{N-1}, \ldots, f_{\bar{N}}) \in \mathcal{E}^{N-1-\bar{N}}$, *we have*

$$B = \left\{ g^N(a, (f_{N-1}, \ldots, f_{\bar{N}}, e_{\bar{N}-1}, \ldots, e_0), \tilde{z}) | (f_{N-1}, \ldots, f_{\bar{N}}) \in \mathcal{E}^{N-1-\bar{N}}, a \in A \right\}$$

then $\mu\left(Conv(B)\right) \leq \Gamma \times \mu\left(Conv(A)\right)$.

Proof.

Note that $Conv(B) = [min(B), max(B)]$. Take $a' = max(A)$ and $a = min(A)$, then $\mu(Conv(A)) = a' - a$ and

$$B \subset [g^N(a, (g_{N-1}, \ldots, g_{\bar{N}}, e_{\bar{N}-1}, \ldots, e_0), \tilde{z}); g^N(a', (f_{N-1}, \ldots, f_{\bar{N}}, e_{\bar{N}-1}, \ldots, e_0), \tilde{z})]$$

for some $(f_{N-1}, \ldots, f_{\bar{N}}) \in \mathcal{E}^{N-1-\bar{N}}, (g_{N-1}, \ldots, g_{\bar{N}}) \in \mathcal{E}^{N-1-\bar{N}}$, such that

$$\mu(Conv(B)) \leq g^N(a', (f_{N-1}, \ldots, f_{\bar{N}}, e_{\bar{N}-1}, \ldots, e_0), \tilde{z}) - g^N(a, (g_{N-1}, \ldots, g_{\bar{N}}, e_{\bar{N}-1}, \ldots, e_0), \tilde{z})$$

Applying (37), one finds $\mu(Conv(B)) \leq \Gamma \times \mu(Conv(A))$.

■

B.1 Step 1

For any $N-$economy, in any period, for $e^N \in \mathcal{E}^N$, define for $1 \leq k \leq N$

$$A_k^{(N)}(e^N) = \{a((f_{N-1}, ..f_k, e_{N-1}, \ldots, e_{N-1-(k-1)}), \tilde{z}_{-[(N-1)-(k-1)]})|(f_{N-1}, ..f_k) \in \mathcal{E}^{N-k}\}$$

In words, $A_k^{(N)}(e^N)$ is the set of wealth values of all agents having a k period history of idiosyncratic shock $(e_{N-1}, \ldots, e_{N-1-(k-1)})$, starting N period ago and finishing $N-k$ periods ago. Note that

$$A_{N-1}^{(N)}(e^N) = \{a((f, e_{N-1}, \ldots, e_1), \tilde{z}_{-1})|f \in \mathcal{E}\} \tag{65}$$

is the set of wealth level of agents who can have the history $e^N \in \mathcal{E}^N$ in the current period, after the idiosyncratic shock is revealed. As a consequence, if $e^N \succeq \tilde{e}^N$, then $a(\tilde{e}^N) \in A_{N-1}^{(N)}(e^N)$.

We have

$$A_{k+1}^{(N)}(e^N) \subset \{g_a(a, (f_{N-1}, ..f_{k+1}, e_{N-1}, \ldots, e_{N-1-(k-1)}, e_{N-k-1}), \tilde{z}_{-[(N-1)-(k-1)]+1}) \\ |(f_{N-1}, ..f_{k+1}) \in \mathcal{E}^{N-k-1}, a \in A_k^{(N)}(e^N)\}.$$

In words, the the set $(A_{k+1}^{(N)}(e^N))$ of wealth values of all agents having a $k+1$ period history of idiosyncratic shock $(e_{N-1}, \ldots, e_{N-k+1})$, (starting N period ago and finishing $N-k+1$ periods ago), is the set of set of wealth values chosen by all agents having a k period history of idiosyncratic shock $(e_{N-1}, \dots, e_{N-k})$, (starting N period ago and finishing $N-k$ periods ago), experiencing a value e_{N-k+1} of their idiosyncratic shock $N-k+1$ periods ago.

If $\bar{N} \leq k < N$, we deduce, applying Lemma (4)

$$\mu\left(Conv(A_{k+1}^{(N)}(e^N))\right) \leq \Gamma\mu\left(Conv(A_k^{(N)}(e^N))\right)$$

Iterating from $k = \bar{N}$ to $k = N-1$, one finds

$$\mu\left(Conv(A_{N-1}^{(N)}(e^N))\right) \leq \Gamma^{N-1-\bar{N}}\mu\left(Conv(A_{\bar{N}}^{(N)}(e^N))\right) \leq \Gamma^{N-\bar{N}-1}[a^{max} - \bar{a}]$$

The last inequality is obtained as $[a^{max} - \bar{a}]$ is the highest set of possible wealth levels.

B.2 Step 2

Recall that

$$\tilde{a}(e^N) = \sum_{\tilde{e}^N \in \mathcal{E}^N} \frac{S(\tilde{e}^N)}{S'(e^N)}\Pi\left(\tilde{e}^N, e^N, \tilde{z}\right)a(\tilde{e}^N), \text{ for all } e^N. \tag{66}$$

Using the the definition of $\Pi\left(\tilde{e}^N, e^N, \tilde{z}\right)$ given in (8), $\tilde{a}(e^N)$ can be written as

$$\tilde{a}(e^N) = \sum_{f \in \mathcal{E}} \frac{S((f, e_{N-1}, \dots, e_1))}{S'((e_{N-1}, \dots, e_0))}M_{e_1,e_0}(s_t)\, a((f, e_{N-2}, \dots, e_1), s_{-1}), \text{ for all } e^N \in \mathcal{E}^N \tag{67}$$

Using (33), one finds $\sum_{f \in \mathcal{E}} \frac{S((f,e_{N-1},\dots,e_1))}{S'((e_{N-1},\dots,e_0))}M_{e_1,e_0}(s_t) = 1$ for all $e^N \in \mathcal{E}^N$. As a consequence, the expression (33) shows that $\tilde{a}(e^N)$ is a weighted sum (with positive weights summing to one), of elements of $A_{N-1}^{(N)}(e^N)$, which is defined by (65). As a consequence, $\tilde{a}(e^N) \in Conv(A_{N-1}^{(N)}(e^N))$.

In addition whatever $\tilde{e}^N \in \mathcal{E}^N$, $a(\tilde{e}^N) \in A_{N-1}^{(N)}(e^N) \subset Conv(A_{N-1}^{(N)}(e^N))$, if $.e^N \succeq \tilde{e}^N$, as a consequence

$$|\tilde{a}(e^N) - a(\tilde{e}^N)| \leq \mu\left(Conv(A_{N-1}^{(N)}(e^N))\right) \leq \Gamma^{N-\bar{N}-1}[a^{max} - \bar{a}]$$

From that we deduce, that

$$lim_{N\to\infty}|T_N(e^{N+1})| = 0 \text{ for all } e^{N+1} \in \mathcal{E}^{N+1}$$

C Proof of Lemma 3

There are two steps in our simplification –that follows Acikgoz (2013). First, we substitute some expressions: after-pooling savings $\tilde{a}$ using (45), capital using (49). Second, we express the Lagrangian and include the individual consumption Euler equation (41) and credit constraints (44).

The Ramsey program (38)–(50) can be rewritten as

$$\begin{aligned}
\max_{\left(r_t,w_t,B_t,(a_{t,e^N},c_{t,e^N},l_{t,e^N})_{e^N\in\mathcal{E}^N}\right)} & \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{e^N\in\mathcal{E}^N}S_{t,e^N}U(c_{t,e^N},l_{t,e^N}) \qquad (68)\\
& -\mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\sum_{e\in\mathcal{E}^N}S_{t,e^N}\lambda_{t,e^N}\Big(U_c(c_{t,e^N},l_{t,e^N})+\nu_{t,e^N}\\
& \qquad -\beta\mathbb{E}_t\left[\sum_{\hat{e}^N\in\mathcal{E}^N}\Pi_{t+1,e^N,\hat{e}^N}U_c(c_{t,e^N},l_{t,e^N})(1+r_{t+1})\right]\Big)
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \qquad B_t + F(A_{t-1}-B_{t-1},L_t) &= G_t + (1+r_t)B_{t-1} \qquad (69)\\
&\quad + r_t(A_{t-1}-B_{t-1}) + w_tL_t\\
a_{t,e^N} + c_{t,e^N} &= w_t n_{e^N} l_{t,e^N} 1_{e_t^N>0} + \delta 1_{e_t^N=0} \qquad (70)\\
&\quad + (1+r_t)\sum_{\hat{e}\in\mathcal{E}^N}\frac{S_{t-1,\hat{e}^N}}{S_{t,e^N}}\Pi_{t,\hat{e}^N,e^N}a_{t-1,\hat{e}^N}\\
l_{t,e^N} &= (w_t n_{e_t^N})^{\varphi}. \qquad (71)
\end{aligned}$$

The Ramsey problem simplifies into maximizing the expression (68) subject to the government budget constraint (69) and individual budget constraints (70) –that should hold for any $e^N \in \mathcal{E}^N$. Equality (71) is the Euler equation determining the individual labor supply.

The proof boils now down to simplifying the following expression

$$
\begin{aligned}
J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} U(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \\
-\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \lambda_{t,e^N} \Big(U_c \left(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi} \right) + \nu_{t,e^N} \\
- \beta \mathbb{E}_t \left[\sum_{\hat{e}^N \in \mathcal{E}^N} \Pi_{t+1,e^N,\hat{e}^N} U_c \left(c_{t+1,\hat{e}^N}, (w_t n_{\hat{e}_t^N})^{\varphi} \right) (1 + r_{t+1}) \right] \Big)
\end{aligned}
$$

Distributing the sum in λ_{t,e^N}, we obtain:

$$
\begin{aligned}
J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \left(U(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) - \lambda_{t,e^N} U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \right) \\
-\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} \sum_{\hat{e}^N \in \mathcal{E}^N} \left(\sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \lambda_{t,e^N} \Pi_{t+1,e^N,\hat{e}^N} \right) U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi})(1 + r_{t+1})
\end{aligned}
$$

We remind that

$$
\Lambda_{t-1,e^N} = \frac{\sum_{\hat{e}^N \in \mathcal{E}^N} S_{t-1,\hat{e}^N} \lambda_{t-1,\hat{e}^N} \Pi_{t,e^N,\hat{e}^N}}{S_{t,e^N}},
$$

which implies

$$
\begin{aligned}
J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \left(U(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) - \lambda_{t,e^N} U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \right) \\
-\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} \sum_{e^N \in \mathcal{E}^N} S_{t+1,e^N} \Lambda_{t+1,e^N} U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi})(1 + r_{t+1})
\end{aligned}
$$

Remarking that $\Lambda_{-1,e^N} = 0$, we have

$$
\begin{aligned}
J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \Big(U(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \\
+ \left(\Lambda_{t-1,e^N} (1 + r_t) - \lambda_{t,e^N} \right) U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \Big)
\end{aligned}
$$

D Proof of Proposition2

We compute the first-order conditions of the simplified Ramsey program (52)–(55). Denoting $\beta^t m(s^t) \mu_t$ the Lagrange multiplier associated to the government budget constraint, we obtain the following expression for the Lagrangian:

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \Big(& U(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \\
& + U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N} \right) \Big) \\
& - \mu_t \beta^t \left(G_t + B_{t-1} + r_t A_{t-1} + w_t L_t - B_t - F(A_{t-1} - B_{t-1}, L_t) \right),
\end{aligned} \tag{72}
$$

where $c_{t,e^N} = (w_t n_{e_t^N})^{\varphi+1} + \delta 1_{e_t^N=0} - a_{t,e^N} + (1+r_t) \sum_{\hat{e}^N \in \mathcal{E}^N} \frac{S_{t-1,\hat{e}^N}}{S_{t,e^N}} \Pi_{t,\hat{e}^N,e^N} a_{t-1,\hat{e}^N}$.

We use the ψ notation defined in (60) that we recall here for the sake of simplicity

$$
\psi_{t,e_t^N}(c_{t,e^N}) = U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) + U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N} \right) \tag{73}
$$

Derivative with respect to a_{t,e^N}. We compute the derivative of the Lagrangian (72) wrt a_{t,e^N}. Note that c_{t,e^N} obviously depends on a_{t,e^N} but this is also the case of any $c_{t+1,\tilde{e}^N}$ where $s^{t+1} \succeq s^t$ and $\tilde{e}^N \in \mathcal{E}^N$. We obtain:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a_{t,e^N}} = & \beta^t S_{t,e^N} \frac{\partial c_{t,e^N}}{\partial a_{t,e^N}} \left(U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) + U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N} \right) \right) \\
& + \mathbb{E}_t \beta^{t+1} \sum_{\tilde{e}^N \in \mathcal{E}^N} S_{t+1,\tilde{e}^N} \frac{\partial c_{t+1,\tilde{e}^N}}{\partial a_{t,e^N}} \\
& \times \left(U_c(c_{t+1,e^N}, (w_{t+1} n_{e_{t+1}^N})^{\varphi}) + U_{cc}(c_{t+1,e^N}, (w_{t+1} n_{e_{t+1}^N})^{\varphi} \left(\Lambda_{t,\tilde{e}^N}(1+r_{t+1}) - \lambda_{t+1,\tilde{e}^N} \right) \right) \\
& - \mathbb{E}_t \beta^{t+1} S_{t,e^N} \mu_{t+1} (r_{t+1} - F_K(A_t - B_t, L_t)),
\end{aligned}
$$

where we remind that $A_t = \sum_{e \in \mathcal{E}^N} S_{t,e^N} a_{t,e^N}$.

Noting that

$$
\begin{aligned}
\frac{\partial c_{t,e^N}}{\partial a_{t,e^N}} &= -1, \\
\frac{\partial c_{t+1,\tilde{e}^N}}{\partial a_{t,e^N}} &= (1+r_{t+1}) \frac{S_{t,e^N}}{S_{t+1,\tilde{e}^N}} \Pi_{t+1,e^N,\tilde{e}^N},
\end{aligned}
$$

we deduce that $\frac{\partial \mathcal{L}}{\partial a_{t+1e}} = 0$ iff, using the ψ notation in (73):

$$\psi_{t,e_t^N}(c_{t,e^N}) = \beta \mathbb{E}_t \sum_{\tilde{e}^N \in \mathcal{E}^N} (1 + r_{t+1}) \Pi_{t+1,e^N,\tilde{e}^N} \psi_{t+1,e_{t+1}^N}(c_{t+1,e^N})$$
$$- \beta \mathbb{E}_t \mu_{t+1}(r_{t+1} - F_K(A_t - B_t, L_t)),$$

This equation can be simplified using the consumption Euler equation (41) stating that

$$U_c(c_{t,e^N}, (w_t n_{e_t^N})^\varphi) + \nu_{t,e^N} = \beta \mathbb{E}_t \left[\sum_{\hat{e}^N \in \mathcal{E}^N} \Pi_{t+1,e^N,\hat{e}^N} U_c \left(c_{t+1,e^N}, (w_{t+1} n_{e_{t+1}^N})^\varphi \right) (1 + r_{t+1}) \right]$$

We obtain

$$U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^\varphi) \left(\Lambda_{t-1,e^N}(1 + r_t) - \lambda_{t,e^N} \right) + \beta \mathbb{E}_t \left[\mu_{t+1}(r_{t+1} - F_K(A_t - B_t, L_t)) \right]$$
$$= \beta \mathbb{E}_t \sum_{\tilde{e}^N \in \mathcal{E}^N} (1 + r_{t+1}) \Pi_{t+1,e^N,\tilde{e}^N}$$

which is the modified Euler equation.

Derivative with respect to B_t. Computing the derivative of the Lagrangian (72) with respect to B_t yields

$$\frac{\partial \mathcal{L}}{\partial B_t} = \beta^t \mu_t - \beta^{t+1} \mathbb{E}_t \left[\mu_{t+1} \left(F_K(A_t - B_t, L_{t+1}) + 1 \right) \right]$$

We deduce the Euler equation for μ_t

$$\mu_t = \beta \mathbb{E}_t \left[\mu_{t+1} \left(F_K(A_t - B_t, L_{t+1}) + 1 \right) \right]. \tag{74}$$

Derivative with respect to r_t. Computing the derivative of the Lagrangian (72) with respect to r_t yields

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r_t} &= \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \Lambda_{t-1,e^N} \\
&+ \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \frac{\partial c_{t,e^N}}{\partial r_t} \left(U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) + U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N} \right) \right) \\
&- \mu_t \beta^t A_{t-1}
\end{aligned}
$$

Noting that

$$
\frac{\partial c_{t,e^N}}{\partial r_t} = \sum_{\hat{e}^N \in \mathcal{E}^N} \frac{S_{t-1,\hat{e}^N}}{S_{t,e^N}} \Pi_{t,\hat{e}^N,e^N} a_{t-1,\hat{e}^N},
$$

we obtain that $\frac{\partial \mathcal{L}}{\partial r_t} = 0$ iff:

$$
\begin{aligned}
\mu_t A_{t-1} = &\sum_{e \in \mathcal{E}^N} S_{t,e^N} U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \Lambda_{t-1,e^N} \\
&+ \sum_{e \in \mathcal{E}^N} \left(\sum_{\hat{e}^N \in \mathcal{E}^N} S_{t-1,\hat{e}^N} \Pi_{t,\hat{e}^N,e^N} a_{t-1,\hat{e}^N} \right) \psi_{t,e_t^N}(c_{t,e^N})
\end{aligned}
$$

Derivative with respect to w_t. Computing the derivative of the Lagrangian (72) with respect to w_t yields:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial w_t} &= \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \left(\frac{\partial c_{t,e^N}}{\partial w_t} \left(U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) + U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N} \right) \right) \right) \\
&+ \beta^t \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \left(\frac{\partial l_{t,e^N}}{\partial w_t} \left(U_l(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) + U_{cl}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \left(\Lambda_{t-1,e^N}(1+r_t) - \lambda_{t,e^N} \right) \right) \right) \\
&- \beta^t \mu_t \left(L_t + w_t L'_{t,w} - F_L(K_{t-1}, L_t) L'_{t,w} \right)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\frac{\partial c_{t,e^N}}{\partial w_t} &= (w_t n_{e_t^N})^{\varphi} n_{e_t^N}, \\
\frac{\partial l_{t,e^N}}{\partial w_t} &= \frac{\varphi}{w_t}(w_t n_{e_t^N})^{\varphi}, \\
U_l(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) &= -(w_t n_{e_t^N}) U_c(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \\
U_{cl}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) &= -(w_t n_{e_t^N}) U_{cc}(c_{t,e^N}, (w_t n_{e_t^N})^{\varphi}) \\
\frac{\partial L_t}{\partial w_t} &= \frac{\varphi}{w_t} \sum_{e^N \in \mathcal{E}^N} n_{e_t^N} S_{t,e^N} l_{t,e^N} = \frac{\varphi}{w_t} L_t
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mu_t \left(L_t + w_t L'_{t,w} - F_L(K_{t-1}, L_t) L'_{t,w} \right) = & \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} n_{e_t^N} (w_t n_{e_t^N})^{\varphi} \psi_{t,e_t^N}(c_{t,e^N}) \\
& - \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} \frac{\varphi}{w_t} (w_t n_{e_t^N})^{\varphi+1} \psi_{t,e_t^N}(c_{t,e^N})
\end{aligned}
$$

and

$$
\mu_t \left(L_t + \varphi L_t - \frac{\varphi}{w_t} F_L(K_{t-1}, L_t) L_t \right) = (1-\varphi) \sum_{e^N \in \mathcal{E}^N} S_{t,e^N} n_{e_t^N} (w_t n_{e_t^N})^{\varphi} \psi_{t,e_t^N}(c_{t,e^N}).
$$

E Structure of the simple model with $N = 1$

The dynamics of households consumption is determined by the three equations

$$
u'(c_{t,e}) = \mathbb{E}_t \left(\alpha_t u'(c_{t+1,e}) + (1-\alpha_t) u'(c_{t+1,u}) \right) (1 + r_{t+1}) \tag{75}
$$

$$
a_{t+1,e} + c_{t,e} = w_t + (1 + r_t) \alpha_t S_{t-1,e} a_{t,e} / S_{t,e}, \tag{76}
$$

$$
c_{t,u} = \delta + (1 + r_{t-1})(1 - \alpha_t) S_{t-1,e} a_{t,e} / S_{t,u}, \tag{77}
$$

The dynamic of public debt is

$$G_t + r_t K_{t-1} + w_t L_t + (1+r_t)B_{t-1} = F(K_{t-1}, L_t, s_{t-1}) + B_t. \quad (78)$$

$$\tilde{r}_t = F_K(K_{t-1}, L_t, s_{t-1}), \quad (79)$$

$$\tilde{w}_t = F_L(K_{t-1}, L_t, s_{t-1}). \quad (80)$$

$$L_t = S_{t,e} n(e) \left(n(e) w_t\right)^{\varphi} \quad (81)$$

$$S_{t,e} a_{t+1,e} = B_t + K_t \quad (82)$$

$$\tau_t^K = 1 - \frac{r_t}{\tilde{r}_t} \quad (83)$$

$$\tau_t^L = 1 - \frac{w_t}{\tilde{w}_t} \quad (84)$$

$$S_{t+1,e} = \alpha_t S_{t,e} + (1-\rho_t) S_{t,u} \quad (85)$$

$$S_{t+1,u} = (1-\alpha_t) S_{t,e} + \rho_t S_{t,u} \quad (86)$$

Before characterizing the optimal fiscal policy in this environment, we start with noting that $\lambda_t(u) = 0$, since credit constraints bind for unemployed agents. The optimal fiscal policy is characterized by the following equations

$$\begin{aligned} u''(c_{t,e})\left((1+r_t)\Lambda_{t,e} - \lambda_{t,e}\right)\lambda_{t,e} = {} & \beta \mathbb{E}_t \alpha_t (1+r_{t+1}) u''(c_{t+1,e}) \quad (87) \\ & \times \left((1+r_{t+1})\Lambda_{t+1,e} - \lambda_{t+1,e}\right) \\ & + \beta \mathbb{E}_t (1-\alpha_t)(1+r_{t+1}) u''(c_{t+1,u})(1+r_{t+1})\Lambda_{t+1,u} \\ & + \beta \mathbb{E}_t \mu_{t+1}\left(F_K(K_t, L_{t+1}, s_t) - r_{t+1}\right) \end{aligned}$$

$$\mu_t = \beta \mathbb{E}_t \mu_{t+1} \left(F_K(K_t, L_{t+1}, s_t) + 1\right) \quad (88)$$

$$\begin{aligned} \mu_t S_{t,e} a_{t+1,e} = {} & S_{t,e} u'(c_{t,e}) \Lambda_{t,e} + S_{t,u} u'(c_{t,u}) \Lambda_{t,u} \quad (89) \\ & + \alpha_{t-1} a_{t,e} S_{t,e} \left(u'(c_{t,e}) + u''(c_{t,e})\left((1+r_t)\Lambda_{t,e} - \lambda_{t,e}\right)\right) \quad (90) \\ & + (1-\rho_{t-1}) a_{t,e} S_{t,u} \left(u'(c_{t,u}) + u''(c_{t,u})(1+r_t)\Lambda_{t,u}\right) \quad (91) \end{aligned}$$

$$\mu_t \left(1 + \varphi \frac{w_t - F_N(K_{t-1}, L_t, s_{t-1})}{w_t}\right) L_t = S_{t,e} n(e) \left(u'(c_{t,e}) + u''(c_{t,e})\left((1+r_t)\Lambda_{t,e} - \lambda_{t,e}\right)\right) \quad (92)$$

$$\Lambda_{t,e} = \alpha_t \lambda_{t-1,e} S_{t-1,e} / S_{t,e} \quad (93)$$

$$\Lambda_{t,u} = (1-\rho_t) \lambda_{t-1,e} S_{t-1,e} / S_{t,u} \quad (94)$$

The condition for credit constraints to be binding for unemployed households is

$$u'(c_{t,u}) > \mathbb{E}_t\left[\left(\left(1-\alpha_t\right)u'(c_{t+1,e}) + \rho_t u'(c_{t+1,u})\right)(1+r_{t+1})\right] \tag{95}$$

For the law of motion the aggregate state described by equations (61), (63) and (64) and for the processes $\{\alpha_t, \rho_t, \Psi_t, G_t\}$ given by the four equations (62),(**??**),(**??**),(**??**), for given initial capital wealth $a_0(e)$, initial public debt B_{-1}, initial promise $\lambda_{-1}(e)$, state s_{-1}, and initial population shares $S_0(e) = \bar{S}$ and $S_0(u) = 1 - \bar{S}$, an equilibrium is a set of 18 variables $\{c_t(e), c_t(u), a_{t+1}(e), r_t, w_t, G_t, K_t, L_t, B_t, \tilde{r}_t, \tilde{w}_t, \tau_t^K, \tau_t^L, S_t(e), S_t(u), \Lambda_t(e), \lambda_t(e), \mu_t\}$satisfying the 18 equations (75) to (94), and the inequality constraint (95).